Dynamics of Relay Relaxation Oscillators

Subbarao Varigonda and Tryphon T. Georgiou

***Abstract*—Relaxation oscillators can usually be represented as a feedback system with hysteresis. The relay relaxation oscillator consists of relay hysteresis and a linear system in feedback. The objective of this work is to study the existence of periodic orbits and the dynamics of coupled relay oscillators. In particular, we give a complete analysis for the case of unimodal periodic orbits, and illustrate the presence of degenerate and asymmetric orbits. We also discuss how complex orbits can arise from bifurcation of unimodal orbits. Finally, we focus on oscillators with an integrator as the linear component, and study the entrainment under external forcing, and phase locking when such oscillators are coupled in a ring.**

I. INTRODUCTION

RELAXATION oscillators represent a class of models which approximate a variety of physical phenomena, from electronic circuitry to circadian biological clocks [1, pp. 169–173], [2], chemical oscillators [3], and ecological systems [4]. Such oscillators consist basically of a feedback system with two elements: a bistable subsystem and a negative integral action. The typical arrangement is shown in Fig. 1. The bistable element in the forward path is a hysteresis-type nonlinearity. The occupancy of one of its two stable states causes a build-up by the integral action in a direction which, in turn, forces the system into the other stable state, and so on.

The best-known example of a relaxation oscillator is perhaps the van der Pol oscillator ([5]–[7]) described by

$$\epsilon\dot{u} = y - \left(\frac{u^3}{3} - u\right)$$
$$\dot{y} = -u$$

where ϵ is a small parameter. This system can be viewed, as in Fig. 1, where the hysteresis is realized by the "fast" bistable subsystem in the first equation. The van der Pol system was originally used to model a tunnel diode circuit, while analogous models have been used to study synchronization in biological systems [6], [8]. In fact, the same hysteresis-feedback paradigm for an oscillatory system has been extensively used in designing a variety of chemical oscillators [3], [9]. In this paper, we consider *relay relaxation oscillators* which consist of a relay hysteresis (cf. [10, p. 262]) and a linear system in feedback as shown in Fig. 2. Such feedback systems with relay are encountered in a wide range of industrial applications [11]–[15] and, more recently, in designing periodic drug delivery devices [16], [17].

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S. Varigonda is with the Department of Chemical Engineering and Materials Science, University of Minnesota 55455, USA.

T. T. Georgiou is with the Department of Electrical and Computer Engineering, University of Minnesota 55455, USA.

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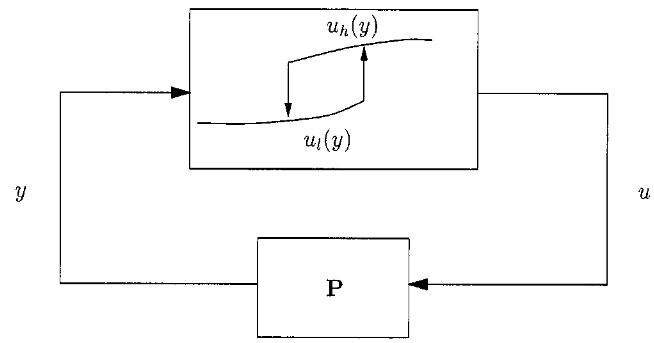


Fig. 1. Feedback system with hysteresis.

Oscillations in hysteretic systems have been extensively studied in the literature. When the hysteresis model has some continuity properties, operator theoretic tools have been particularly successful [18]–[20]. For relay systems, several approaches have been used. More specifically, graphical techniques and harmonic balance methods have been used early on [11], [12], [18], [21, Ch. 7]. These seem more suitable for low-order models and for an approximate analysis. On the other hand, state-space methods and analysis of Poincaré maps allow a more accurate analysis of periodic phenomena. In particular, conditions for the existence and stability of unimodal periodic orbits (i.e., orbits having exactly two relay switchings per period) were obtained for two-dimensional systems in [22, Ch. 8], and for higher dimensional systems and for systems with time-delays in [13] using Poincaré maps. A sufficient condition for the existence of a globally stable unimodal periodic orbit was given in [23] in terms of the transfer function of the linear component. Time domain methods (cf. [14], [24]) and fixed point theorems (cf. [25]) have also been used to obtain sufficient conditions for the existence of unimodal periodic orbits. The relationship between the state-space and frequency-domain approaches is discussed by Astrom [13]. The case of pure relay (i.e., no hysteresis) can be obtained in the limit of the on–off switching points approaching each other. Feedback systems with pure relay have been studied in greater detail by Johansson *et al.* (see [26] and the references therein).

The approach we follow in this paper is based on state-space representations and the analysis of Poincaré maps. In Section II, we consider the system in Fig. 2 with a controllable and observable realization for the linear system *P* and obtain necessary and sufficient conditions for the existence of unimodal periodic orbits. This follows the aforementioned analysis in [13], but provides a more complete picture. We show the existence of a degenerate case with a continuum of periodic orbits. We also show that asymmetric periodic orbits can exist in a system with a symmetric relay. Complex orbits involving many relay switchings

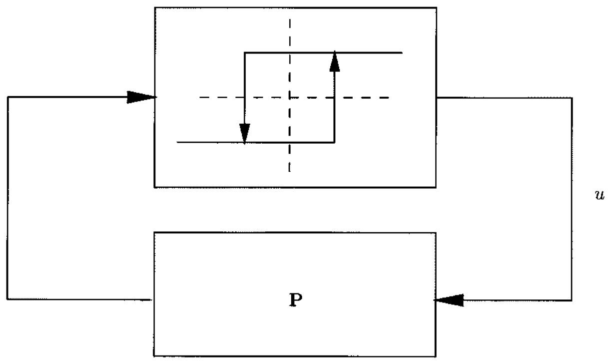


Fig. 2. Feedback connection of relay hysteresis and a linear system.

per period can be obtained from the bifurcation of unimodal orbits.

In Section III, we consider the special case of a relay relaxation oscillator consisting of relay hysteresis and integral feedback. Such a system has a unique globally stable periodic orbit, and resembles the van der Pol oscillator. It also presents the advantage of piece-wise linear maps to study the behavior of driven or coupled systems. We study the entrainment of this oscillator under periodic external forcing at the input of the integrator. In particular, we show the existence of primary and subharmonic entrainment regions in the relevant parametric space, and contrast our results with results from earlier studies on general relaxation oscillators [7], [8], [27]–[29], [6, p. 339]. Finally, we consider rings of relay relaxation oscillators and demonstrate synchrony and phase locking phenomena. The behavior of driven or coupled oscillators is of interest in variety of subjects, such as in biological systems, lasers, phase transitions, reaction-diffusion systems, etc. [8], [29]–[32]. We show results for two and three member rings with both uni and bidirectional coupling and observe that out-of-phase locking is possible even under excitatory coupling (see Remark 9). The analysis can be extended in principle to larger systems.

II. Autonomous Oscillations

Consider the feedback system shown in Fig. 2 where the SISO linear system *P* is described by a controllable and observable state space model

$$\dot{x} = Ax + bu \tag{1}$$
$$y = cx. \tag{2}$$

The relay switches from its *on*-state (output 1) to *off*-state (output -1) if the input reaches -1 from above and switches from the *off*-state to the *on*-state if the input reaches one from below. If the relay is not symmetric with respect to the origin, the analysis will essentially be the same, after an affine transformation. The case when $y = cx + du$, with $d > -1$ for well-posedness, can be reduced to the above form by defining a new output $\hat{y} = (c/d+1)x$ which preserves the states at which switching occurs.

The following analysis involves identifying the switching surfaces in state space that correspond to the switching points of the relay, and defining a Poincaré return map from one of the surfaces to itself. The fixed points of the map then correspond to unimodal periodic orbits after they verify some consistency conditions.

A. Existence of Periodic Orbits

The switching to $u = -1$ and $u = 1$ occurs on the level surfaces S_- and S_+ in $\mathbb{R}^n$ defined by $cx = -1$ and $cx = 1$, respectively. The space $\mathbb{R}^n$ is divided into three regions by S_- and S_+, namely, $R_- = \{x: cx < -1\}$, $R_\pm = \{x: cx > -1, cx < 1\}$, and $R_+ = \{x: cx > 1\}$, as shown in Fig. 3. Depending on the state, the system is governed by one of the two models M_- and M_+ defined by

$$M_-: \dot{x} = Ax - b, \qquad y = cx \tag{3}$$
$$M_+: \dot{x} = Ax + b, \qquad y = cx. \tag{4}$$

If the state is in R_-, the system follows M_- until the trajectory hits S_+, at which point, the input u switches to $+1$. The system then follows M_+ until the trajectory hits S_- and u switches to -1. In $R_\pm$, both M_- and M_+ are allowed depending on the initial state. Notice that crossing S_+ (S_-) has no effect when the system follows M_+ (M_-).

The response of *P* from initial state $x(0)$ to a *constant* input $u = \pm 1$ over $[0, t]$ is given by $x(t) = e^{At}x(0) + \int_0^t e^{A\tau}\, d\tau\, bu$. For convenience, we will write $\int_0^t e^{A\tau}\, d\tau = \sum_{i=1}^{\infty} (A^{i-1}T^i/i!)$ as $A^{-1}(e^{At} - I)$, but it should be noted that this function is well defined even if A does not have an inverse. Hence we have $x(t) = e^{At}x(0) + A^{-1}(e^{At} - I)bu$. The state transition maps for M_- ($u = -1$) and M_+ ($u = 1$) are given by $\phi_-(t, 0, x_0) = e^{At}x(0) - A^{-1}(e^{At} - I)b$ and $\phi_+(t, 0, x_0) = e^{At}x(0) + A^{-1}(e^{At} - I)b$, respectively.

Consider a trajectory that has at least a finite number of switchings (see Fig. 3). Let ξ_k be the point on the trajectory where it switches to M_+ i.e., $\xi_k \in S_+$ and $c\xi_k = 1$. Suppose, after time t_k, the trajectory hits S_- at η_k ($c\eta_k = -1$) and switches to M_-. Let τ_k be the time taken for the trajectory to hit S_+ again at ξ_{k+1}. Suppose that the trajectory is transversal to the switching surfaces at ξ_k and η_k.

We can then define the two Poincaré maps $\eta_k = P_+(\xi_k)$ and $\xi_{k+1} = P_-(\eta_k)$ as follows:

$$\eta_k = e^{At_k}\xi_k + A^{-1}(e^{At_k} - I)b \tag{5}$$
$$\xi_{k+1} = e^{A\tau_k}\eta_k - A^{-1}(e^{A\tau_k} - I)b \tag{6}$$

where t_k and τ_k are implicitly determined as the smallest positive solutions of $c\eta_k = -1$ and $c\xi_{k+1} = 1$, respectively. The composition $P = P_- \circ P_+$, which maps S_+ to itself,defines a Poincaré return map for the feedback system. Similarly, $\tilde{P} = P_+ \circ P_-$,which maps S_- to itself, is also a Poincaré return map. Clearly, the fixed points and periodic points of a return map correspond to the periodic orbits of the feedback system. Complex attractors of the map correspond to complex oscillations of the system. The stability properties are inherited as well. We call the periodic orbits that have exactly two switchings of u in each period, *unimodal orbits* and they correspond to the fixed points of the Poincaré map.

Remark 1: The maps P_- and P_+ inherit a symmetry from the underlying models M_- and M_+, despite the presence of the implicitly determined parameters t_k and τ_k. Namely, $P_-(\eta) =$

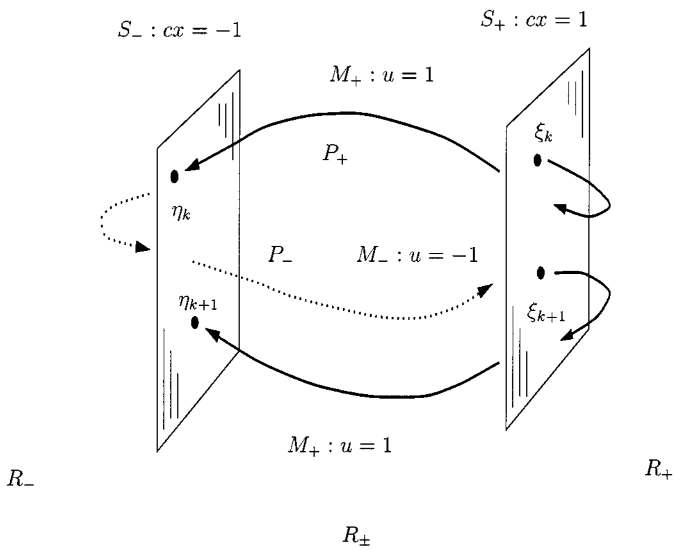


Fig. 3. Partitioning of the state space and the Poincaré maps.

$-P_+(-\eta)$. This implies that the return maps satisfy $P(\xi) = (P_- \circ P_+)(\xi) = (-P_+)^2(\xi)$ and $\tilde{P}(\eta) = (P_+ \circ P_-)(\eta) = (-P_-)^2(\eta)$. If ξ is a fixed point of $P = (-P_+)^2$, so is $-\eta$ where $\eta = P_+(\xi)$. Thus, the fixed points of P are either the fixed points of $-P_+$, in which case they satisfy $\xi + \eta = 0$, or they come in pairs ξ and $-\eta$. A similar conclusion holds for $\tilde{P}$. Hence, computing the fixed points of the map $-P_+(\xi)$ or $-P_-(\eta)$, as was done in [13], may not yield all the unimodal-peridic orbits. Other unimodal orbits that are period-two (but not period-one) fixed points of this map can exist (shown in Sections II-B and -D) and hence, we need to analyze the fixed points of the second iterate.

In the subsequent analysis, we will use the return map $\xi_{k+1} = P(\xi_k) = P_-(P_+(\xi_k))$. The following lemma is useful in determining the fixed points. Let A', $\mathcal{N}(A)$, $\mathcal{R}(A)$ and spec(A) denote the transpose, null space, range and spectrum of A, respectively. We say that a scalar function $f(s)$ is *defined on the spectrum of a matrix* A when $f(s)$ and its derivatives upto the required order (depending on the Jordan structure of A) are defined at the eigenvalues of A (see [33]).

Lemma 1: Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ with (A, b) controllable, and let f, g be functions defined on the spectrum of A. Let $f_0(s)$, $g_0(s)$ be interpolation polynomials of f, g, respectively (i.e., such that $f(A) = f_0(A)$ and $g(A) = g_0(A)$), and let $r(s)$ be their greatest common divisor and p, q be defined by $f_0(s) = r(s)p(s)$ and $g_0(s) = r(s)q(s)$, respectively. The equation

$$f(A)\xi = g(A)b \tag{7}$$

has a solution for ξ if and only if

$$\mathcal{R}(q(A)) \subseteq \mathcal{R}(p(A)) + \mathcal{N}(r(A)). \tag{8}$$

Proof: If (7) holds, then, by multiplying from the left with powers of A, we get

$$f(A)[\xi \quad A\xi \quad \cdots \quad A^{n-1}\xi] = g(A)\,[b \quad Ab \quad \cdots \quad A^{n-1}b]. \tag{9}$$

From the invertibility of $[b \quad Ab \cdots A^{n-1}b]$, it follows that the equation

$$f(A)\xi_0 = g(A)v \tag{10}$$

has a solution ξ_0 for all $v \in \mathbb{R}^n$. Hence, $\forall v \in \mathbb{R}^n$, $\exists \xi_0$ such that

$$r(A)[p(A)\xi_0 - q(A)v] = 0 \tag{11}$$

which implies that $\forall v$, $\exists w \in \mathcal{N}(r(A))$ and ξ_0 such that $q(A)v = p(A)\xi_0 + w$. Therefore, $\mathcal{R}(q(A))$ is contained in $\mathcal{R}(p(A)) + \mathcal{N}(r(A))$.

For the converse, assume that (8) is satisfied. Since $q(A)b \in \mathcal{R}(q(A))$, $\exists w \in \mathcal{N}(r(A))$ and ξ such that $q(A)b = p(A)\xi + w$. Multiplication by $r(A)$ gives (7). ■

Corollary 1: $(I - e^{2AT})\xi = A^{-1}(e^{AT} - I)^2 b$ where T is a nonzero scalar has a solution if and only if $e^{AT} + I$ is nonsingular.

Proof: Applying *Lemma 1*, with $f(A) = I - e^{2AT}$ and $g(A) = A^{-1}(e^{AT} - I)^2$, we observe that $r(A) = I - e^{AT}$, $p(A) = e^{AT} + I$ and $q(A) = A^{-1}(I - e^{AT})$ with interpolation polynomials implied. Notice that $p(s)$ and $q(s)$ are coprime as they do not have a common zero. By *Lemma 1*, a solution exists iff

$$\mathcal{R}(A^{-1}(I - e^{AT})) \subseteq \mathcal{R}(e^{AT} + I) + \mathcal{N}(I - e^{AT}). \tag{12}$$

However, $\mathcal{N}(I - e^{AT}) \subseteq \mathcal{R}(e^{AT} + I)$ since $(I - e^{AT})v = 0$ implies that $(e^{AT} + I)v = 2v$ for any v. Hence, (12) reduces to

$$\mathcal{R}(A^{-1}(I - e^{AT})) \subseteq \mathcal{R}(e^{AT} + I). \tag{13}$$

This condition is trivially satisfied when $e^{AT} + I$ is nonsingular. We now claim that (13) cannot be satisfied when $e^{AT} + I$ is singular. To see this, let us rewrite (13) as

$$\mathcal{N}(e^{A'T} + I) \subseteq \mathcal{N}({A'}^{-1}(I - e^{A'T})). \tag{14}$$

Suppose $v \neq 0$ and $v \in \mathcal{N}(e^{A'T} + I)$, i.e.,

$$e^{A'T}v = -v. \tag{15}$$

Equation (14) then implies that ${A'}^{-1}(I - e^{A'T})v = 0$ which, on multiplication by A', gives $(I - e^{A'T})v = 0$. Hence, we have

$$e^{A'T}v = v. \tag{16}$$

Equations (15) and (16) imply that $v = 0$, which is a contradiction. Thus, $e^{AT} + I$ must be nonsingular for a solution to exist. ■

A unimodal-periodic solution of the feedback system can, in general, have different time intervals between successive switches. We first consider the case of equal intervals between switches, whereas the general case is discussed in Section II-D. The following result was first stated by Astrom [12], but a complete proof is provided here as it reveals the existence of nonisolated-periodic orbits, which have not been identified earlier.

Proposition 1: The system in Fig. 2, where *P* is described by (1), (2), has a unimodal-periodic orbit of half-period T, iff, $\chi(T) = 0$ and $\psi(t) > 0$ for $0 < t < T$ where

$$\chi(T) := 1 + c(I + e^{AT})^{-1}A^{-1}(e^{AT} - I)b \tag{17}$$

$$\psi(t) := 1 + c[e^{At}\xi_0 + A^{-1}(e^{At} - I)b] \tag{18}$$
$$\xi_0 := -(I + e^{AT})^{-1}A^{-1}(e^{AT} - I)b. \tag{19}$$

Proof: To prove the necessity, suppose there is a periodic orbit with half-period T. Let $\xi \in S_+$ and $\eta \in S_-$ be the states when the switchings to $u = \pm 1$ occur on this periodic orbit. Thus, we have

$$c[\xi \quad \eta] = [1 \quad -1] \tag{20}$$
$$\eta = P_+(\xi) = e^{AT}\xi + A^{-1}(e^{AT} - I)b \tag{21}$$
$$\xi = P_-(\eta) = e^{AT}\eta - A^{-1}(e^{AT} - I)b \tag{22}$$

where we used the fact that the trajectory takes time T to reach η from ξ with $u = 1$, and to reach ξ from η with $u = -1$. Solving for ξ and η from (21) and (22), we get

$$(I - e^{2AT})\xi = A^{-1}(e^{AT} - I)^2 b \tag{23}$$
$$(I - e^{2AT})\eta = -A^{-1}(e^{AT} - I)^2 b. \tag{24}$$

Since (23) has a solution, Corollary 1 implies that $e^{AT} + I$ is nonsingular. Solving for ξ and η from (23) and (24), we get

$$\xi = -(e^{AT} + I)^{-1}A^{-1}(e^{AT} - I)b + w \tag{25}$$
$$\eta = (e^{AT} + I)^{-1}A^{-1}(e^{AT} - I)b + w \tag{26}$$

where $w \in \mathcal{N}(e^{AT} - I)$. Since $c(\xi + \eta) = 0$ from (20), we also have $cw = 0$. Notice that $w = 0$ is always a solution which corresponds to $\xi = \xi_0$. Now, $c\xi = 1$ gives $\chi(T) = 0$ since $\chi(T) = 1 - c\xi = 1 - c\xi_0$.

Since the trajectory from ξ does not hit S_- until T, it remains in the region $cx > -1$ for $0 < t < T$ and we have

$$1 + c\phi_+(0, t, \xi) = \psi(t) + ce^{At}w > 0. \tag{27}$$

Similarly, from the condition $c\phi_-(0, t, \eta) < 1$ for $0 < t < T$, we get

$$1 - c\phi_-(0, t, \eta) = \psi(t) - ce^{At}w > 0. \tag{28}$$

The addition of (27) and (28), gives $\psi(t) > 0$ for $0 < t < T$.

For sufficiency, let $\chi(T) = 0$ and $\psi(t) > 0$ for $0 < t < T$. The former implies that $c\xi_0 = 1$. Hence $\xi_0 \in S_+$ and u must be 1. The trajectory from ξ_0 does not hit S_- before time T since $\psi(t) > 0$ for $0 < t < T$. After time T, it can be shown that the trajectory hits S_- at $\eta_0 := -\xi_0$ and hence, the input switches to -1. Again by a similar argument, the trajectory from η_0 with $u = -1$ hits S_+ at ξ_0 after time T, closing the orbit. Thus, there is a periodic orbit passing through ξ_0.

The periodic orbit in Proposition 1 is in general, neither unique nor isolated. The following example illustrates the case of multiple (but isolated) orbits. An interesting case of a continuum of orbits will be discussed in Section II-B. For a frequency-domain interpretation of this result and an extension to systems with time delay, see [12].

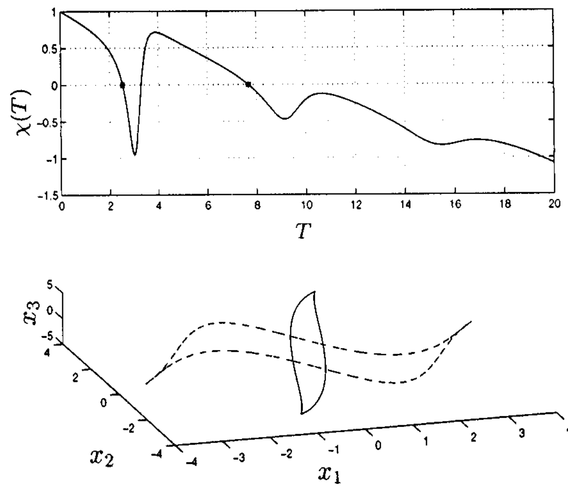


Fig. 4. Graph of $\chi(T)$ with roots T_1 and T_2 marked by asterisks and the state space with the corresponding periodic orbits (solid and dashed) for the system in Example 1.

Example 1: This example shows that the periodic orbit in *Proposition 1*, in general, is not unique. Let

$$A = \begin{bmatrix} -0.05 & 0 & 0 \\ 0 & -0.1 & 1 \\ 0 & -1 & -0.1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad c = \begin{bmatrix} 0.2 \\ 0.2 \\ 0 \end{bmatrix}'.$$

In this case, two different solutions of $\chi(T) = 0$, $T_1 = 2.53$ and $T_2 = 7.67$ satisfy $\psi(t) > 0$ and give rise to two different periodic orbits. The orbit with the smaller period passes through $\xi = (1.2657 \quad 3.7342 \quad 1.8630)'$, and the other through $\xi = (3.7913 \quad 1.2087 \quad 0.2143)'$. The plots of $\chi(T)$ and the periodic orbits in state space are shown in Fig. 4.

B. Degenerate Case

The periodic orbit obtained using Proposition 1 may not, in general, be isolated. Here, we provide a condition that guarantees that the orbit is isolated. We also discuss what happens when this condition is not satisfied.

Proposition 2: A unimodal periodic orbit with half-period T is isolated iff

$$\mathcal{N}(e^{AT} - I) \cap \mathcal{N}(c) = \{0\}. \tag{29}$$

Moreover, there exists a continuum of periodic orbits when (29) is not satisfied.

Proof: The sufficiency of (29) is straight forward. The roots of $\chi(T)$ are isolated and hence any continuum of orbits must have the same half-period. Referring to (25), (26) and the following statement, w must belong to $\mathcal{N}(e^{AT} - I) \cap \mathcal{N}(c)$. Equation (29) then implies that $w \equiv 0$ and hence $\xi = \xi_0$ and $\eta = -\xi_0$ are unique. Hence there is only one periodic orbit with a given T.

Now for the necessity, suppose (29) is violated and $\exists w \neq 0$ in $\mathcal{N}(e^{AT} - I) \cap \mathcal{N}(c)$. We claim that for any such w of sufficiently small magnitude, there is a periodic orbit passing through and switching at $\xi = \xi_0 + w$ and $\eta = -\xi_0 + w$. First observe that $\xi \in S_+$, $\eta \in S_-$, $\eta = \phi_+(T, 0, \xi)$ and $\xi = \phi_-(T, 0, \eta)$. To show that the trajectory through ξ and η is a periodic orbit,

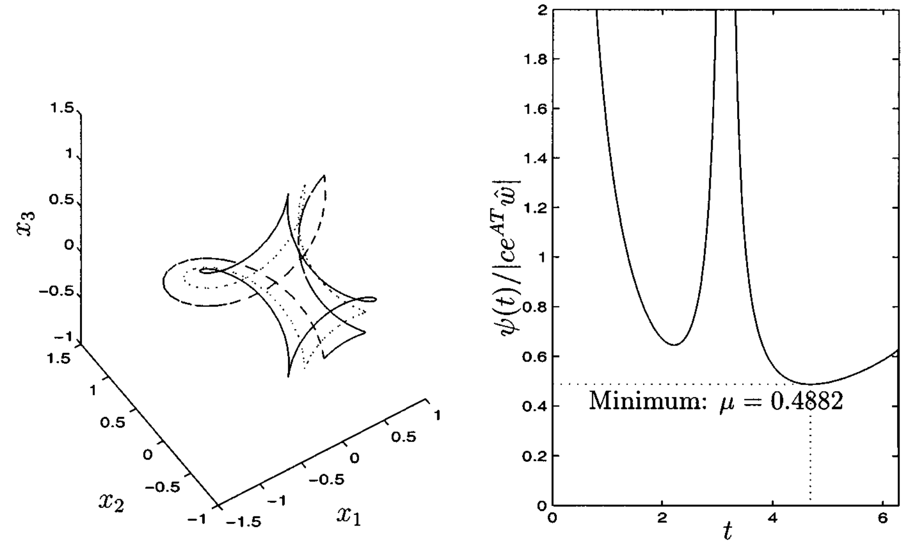


Fig. 5. Periodic orbits with $\|w\| = 0$ (solid), 0.2 (dotted) and 0.438 (dashed) and the plot of $\psi(t)/|ce^{At}\hat{w}|$ for Example 2.

we only need that the switchings do not occur before time T. Since $\psi(t)$ is strictly positive in $(0, T)$, for sufficiently small $\|w\|$, (27), (28) are still satisfied. To compute the bound on $\|w\|$, observe that (27), (28) are equivalent to

$$|ce^{At}w| < \psi(t) \quad \text{for } 0 < t < T. \tag{30}$$

Let $\hat{w}$ be the unit vector along w and

$$\mu := \inf_{0<t<T} \frac{\psi(t)}{|ce^{At}\hat{w}|}. \tag{31}$$

Thus, for $\|w\| < \mu$, (27) and (28) are satisfied, and every ξ with $\|w\| < \mu$ has a periodic orbit passing through it. ■

Remark 2: The matrix $e^{AT} - I$ is nonsingular iff spec$(A) \cap \{\lambda: \lambda = (j\pi 2k/T),\ k \in \mathbb{Z}\}$ is empty. Moreover, if A does not have any purely imaginary (nonzero) eigenvalues, $\mathcal{N}(e^{AT} - I) = \mathcal{N}(A)$ and the observability of (A, c) guarantees (29).

Remark 3: The existence of a continuum can also be inferred from *Remark 1* and the symmetry of the inequality condition in *Proposition 1*. If the pair (ξ, η) such that $2w := \xi + \eta \neq 0$ represents a periodic orbit, so is the pair $(-\eta, -\xi)$ and so is each of their convex combinations. All these orbits have the same period.

A degenerate case with a continuum of periodic orbits is structurally unstable in the sense that this behavior is not robust to small perturbations in A, b and c but, is not impossible as illustrated in the following example.

Example 2: Consider the system

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -0.01 \end{bmatrix} \quad b = \begin{bmatrix} -0.3184 \\ -0.3184 \\ -0.3184 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}'.$$

We have $\chi(2\pi) = 0$ and $\psi(t) > 0$ for $0 < t < 2\pi$. Hence, there is a periodic orbit at $T = 2\pi$ passing through $\xi_0 = [0\ \ 0\ \ 1]'$. But (29) is not satisfied. There is a nonzero vector $\hat{w} = [1\ \ 0\ \ 0]'$ in $\mathcal{N}(e^{AT} - I) \cap \mathcal{N}(c)$. From (31), μ can be computed to be 0.4882 (see Fig. 5. The eigenvalues of D_{P_-} are 0, 0.9691 and 1 for the nominal orbit through ξ_0. There is a continuum of periodic orbits through each $\xi = \xi_0 + w$ with $\|w\|; < \mu$, some of which are shown in Fig. 5. When $\|w\| \geq \mu$, there are no more orbits through $\xi = \xi_0 + w$ as the input switches before time T.

C. Stability of Periodic Orbits

The stability of the Poincaré return map $P = P_- \circ P_+$ is equivalent to the stability of the periodic orbit. However, due to the nonlinearity of the Poincaré map, it is hard to guarantee global stability even with a stable A matrix. Some results have been obtained by Kolesov [23] using the notion of *strongly positive* transfer functions. Global stability for a class of two dimensional systems has been shown by Megretski [14]. Recently, a computational procedure to check global stability using quadratic Lyapunov functions, is proposed in [34]. However, linear stability results can be easily obtained.

Let $D_{P_+}|_\xi$, $D_{P_-}|_\eta$ denote the derivatives of P_+ at ξ and P_- at η respectively. From Remark 1, we can see that $D_{P_-}|_{\eta=-\xi} = D_{P_+}|_\xi$. Assuming that the orbit hits the switching surfaces transversally, D_{P_+} can be obtained as

$$D_{P_+}|_{\xi_k} = \left[I - \frac{vc}{cv}\right] e^{At_k} \tag{32}$$

where $v = e^{At_k}(A\xi_k + b) = A\eta_k + b$ is the tangent vector just before the switching. The derivative of the return map $P = P_- \circ P_+$ is then $D_P = D_{P_-}|_{\eta_k} D_{P_+}|_{\xi_k} = D_{P_+}|_{-\eta_k} D_{P_+}|_{\xi_k}$. A fixed point ξ of P is stable if D_P has all eigenvalues within the open unit disk.

Remark 4: It can be noticed that $cD_{P_+} = cD_{P_-} = 0$ and hence, D_{P_+}, D_{P_-} and D_P always have a zero eigenvalue. This is due to the fact that both the domain and range of these maps have been restricted to a hyperplane thus, in essence, making them $n - 1$ dimensional.

Remark 5: When A is stable, it is possible to obtain a sufficient condition for global stability using the contraction mapping principle. The idea involves identifying a compact invariant set $\Omega \subset \mathbb{R}^n$ of the system as in [25], [34] and then requiring that

$\|D_P\| = \|D_{P_+}\|^2 < 1$ on $\Omega \cap S_+$. These results will be presented in a future work.

For an isolated symmetric unimodal periodic orbit, $\xi + \eta = 0$ and we can see that $D_{P_-} = D_{P_+}$ and $v = e^{AT}(I + e^{AT})^{-1}b$ The linear stability result can then be stated as below [12].

Proposition 3: Suppose the system has an isolated symmetric unimodal periodic orbit of half-period T and the transversality condition $cv \neq 0$, where $v := e^{AT}(I + e^{AT})^{-1}b$, is satisfied. Then,

$$D_{P_-} = D_{P_+} = \left[I - \frac{vc}{cv}\right] e^{AT}$$

and the orbit is *locally stable* if all the eigenvalues of D_{P_-} are inside the open unit disk in the complex plane. It is *unstable* if at least one of the eigenvalues of D_{P_-} is outside the unit disk.

Remark 6: When the system exhibits degeneracy, any $w \neq 0$ in $\mathcal{N}(e^{AT} - I) \cap \mathcal{N}(c)$ is an eigenvector of D_{P_+} with a unity eigenvalue, confirming that the orbit is not asymptotically stable.

Example 3: This example shows that even if A is not stable, the periodic orbit may be locally stable. Consider the system

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad c = [-0.425\,50 \;\; -1.7752].$$

This system is unstable. However, $\chi(T)$ has a root at $T = 1.2786$ and $\psi(t)$ stays positive for $0 < t < T$. So, by Proposition 1, there exists a unimodal periodic orbit. It passes through $\xi = [-0.5644 \;\; -0.4281]'$. The orbit also satisfies the transversality condition and

$$D_{P_+} = \begin{bmatrix} 0.2636 & -2.866 \\ -0.063\,19 & 0.6869 \end{bmatrix}.$$

The eigenvalues of D_{P_+} are 0.9505 and 0, and the periodic orbit is locally stable. The system is unbounded when the initial condition is too far from this orbit.

D. Asymmetric Unimodal Oscillations

Using the Poincaré maps, we can also obtain the necessary and sufficient conditions for unimodal periodic orbits with different time intervals between the relay switchings. Conditions for the existence of such asymmetric oscillations for systems with an asymmetric relay have been given in [12], but they have not been identified in any system. Here, we show that asymmetric orbits can exist even in systems with a symmetric relay.

Let T_+ and T_- ($T_+ \neq T_-$) be the time intervals corresponding to $u = 1$ and $u = -1$, respectively. Let $\xi \in S_+$, $\eta \in S_-$ be the states at switching and define $T := (T_+ + T_-)/2$. Then we have

$$\eta = e^{AT_+}\xi + A^{-1}(e^{AT_+} - I)b$$
$$\xi = e^{AT_-}\eta - A^{-1}(e^{AT_-} - I)b$$

which can be written as

$$(I - e^{2AT})\xi = A^{-1}(e^{2AT} - 2e^{AT_-} + I)b \tag{33}$$
$$(I - e^{2AT})\eta = A^{-1}(-e^{2AT} + 2e^{AT_+} - I)b. \tag{34}$$

Adding the two and letting $w := \xi + \eta$, we get

$$(I - e^{2AT})w = 2A^{-1}(e^{AT_+} - e^{AT_-})b. \tag{35}$$

It can be shown, using Lemma 1, that for $T_+ \neq T_-$ there is a solution w iff $I - e^{2AT}$ is nonsingular. In particular, A cannot be singular. Hence, we have

$$\xi = (I - e^{2AT})^{-1}A^{-1}(e^{2AT} - 2e^{AT_-} + I)b \tag{36}$$
$$\eta = (I - e^{2AT})^{-1}A^{-1}(-e^{2AT} + 2e^{AT_+} - I)b. \tag{37}$$

Let

$$\psi_+(t) := 1 + c[e^{At}\xi + A^{-1}(e^{At} - I)b] \tag{38}$$
$$\psi_-(t) := 1 - c[e^{At}\eta - A^{-1}(e^{At} - I)b]. \tag{39}$$

The necessary and sufficient conditions can now be stated as follows.

Proposition 4: The feedback system in Fig. 2 has a unimodal periodic orbit with different time intervals between relay switchings if and only if $\exists T_+ > 0, T_- > 0$ ($T_+ \neq T_-$) satisfying

$$c(I - e^{2AT})^{-1}A^{-1}(e^{2AT} - 2e^{AT_-} + I)b = 1 \tag{40}$$
$$c(I - e^{2AT})^{-1}A^{-1}(e^{AT_+} - e^{AT_-})b = 0 \tag{41}$$

with $T := (T_+ + T_-)/2$, $\psi_+(t) > 0$, and $\psi_-(t) > 0$ for $0 < t < T_+$ and $0 < t < T_-$, respectively.

Moreover, the asymmetric periodic orbit is stable if all the eigenvalues of $D_P = D_{P_+}|_{-\eta} D_{P_+}|_{\xi}$, where D_{P_+} is given by (32), are within the open unit disk. It is unstable if at least one eigenvalue is outside the unit disk.

Proof: Equations (40) and (41) follow directly from $c\xi = 1$, $cw = 0$ and (36) and (37). The positivity of ψ_+ and ψ_- ensures that the input does not switch before T_+ or T_- as applicable in each part of the orbit. The stability analysis of Section II-C applies here as well and the second part of the proposition is straight forward after noting that $D_{P_-}|_{\eta} = D_{P_+}|_{-\eta}$ by symmetry (see Remark 1).

Remark 7: The inequality conditions in Proposition 4 also have a symmetry. Together with the symmetry of the return map in Remark 1, this implies that if there is a periodic orbit through (ξ, η) with periods T_+ and T_-, then there is another orbit through $(-\eta, -\xi)$ with periods T_- and T_+. However, unlike in Remark 3, convex combinations are not periodic orbits.

Example 4: This example shows that unimodal periodic orbits with $T_+ \neq T_-$ can exist. Let

$$A = \begin{bmatrix} -0.05 & 1 & 0 \\ -1 & -0.05 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} -0.9615 \\ 0 \\ -0.9615 \end{bmatrix}'.$$

This system satisfies (40) and (41) for $T_+ = 0.6361$ and $T_- = 8.1404$. We have $\xi = (-1.0985 \;\; 0.2080 \;\; 0.0585)'$ and $\eta = (0.0403 \;\; 1.1878 \;\; 0.9997)'$. The inequalities $\psi_+(t) > 0$ for $0 < t < T_+$ and $\psi_-(t) > 0$ for $0 < t < T_-$ are also satisfied as shown in Fig. 6. Hence, by Proposition 4, there is a unimodal periodic orbit through ξ and η. The eigenvalues of $D_P = D_{P_-}D_{P_+}$ are 0, −0.08 and 1996.8 and hence the orbit is unstable. A simulation over one period $2T = T_+ + T_-$ is shown in Fig. 6.

It is possible to obtain more complex orbits involving many switchings of the relay from the bifurcation of unimodal orbits $(T_+ = T_- = T)$. The bifurcation of the unimodal orbit is caused by the function $\psi(t)$ hitting zero before time T, thus causing an early switching. When there are no other stable unimodal orbits, this results in an orbit with more than two switchings. A similar bifurcation in relation to sliding orbits in relay systems has been reported recently in [35].

The following example illustrates a phenomenon analogous to period doubling observed in smooth nonlinear systems [6, Ch. 4].

Example 5: Consider the relay feedback system with A, b, and c given by the realization of

$$\frac{-80(s-1-j)(s-1+j)(s+2)}{(s+0.3-j)(s+0.3+j)(s+1-pj)(s+1+pj)}.$$

The bifurcation parameter p determines the location of a pair of poles. This linear system is stable and the feedback system has a unique-stable unimodal periodic orbit (see Fig. 7) for $p < p_c$, $(p_c \approx 2.59)$]. Only the root of $\chi(T)$, marked with an asterisk, corresponds to a valid periodic orbit for $p < p_c$. As p increases above p_c, the unimodal orbit ceases to exist since $\psi(t)$ hits zero before T causing an early switch. The system then goes to a periodic orbit with four switchings as shown in Fig. 7.

Increasing p further results in a succession of such bifurcations resulting in orbits with many switchings. The orbit for $p = 15$ with a large number of switchings is shown in Fig. 8.

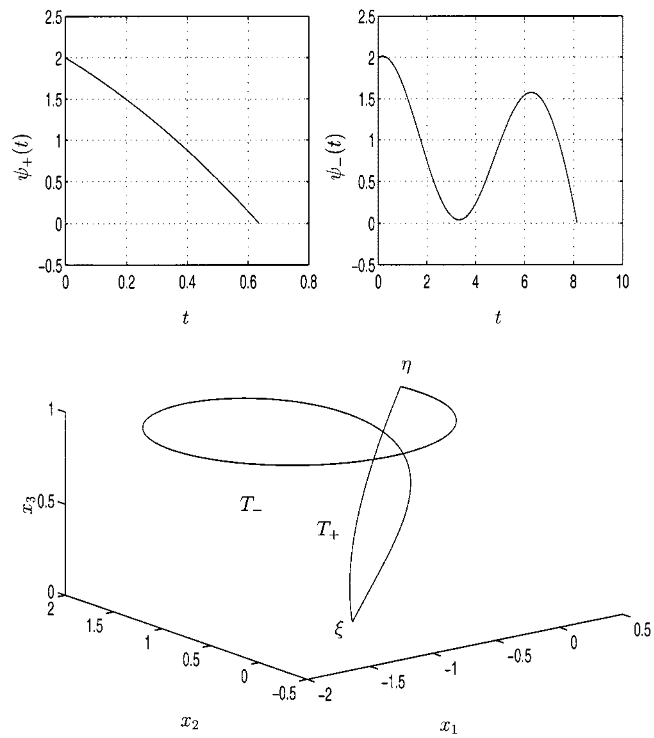


Fig. 6. $\psi_+(t)$, $\psi_-(t)$ and the periodic orbit with $T_+ = 0.6361$ and $T_- = 8.1404$ for the system in *Example 4*.

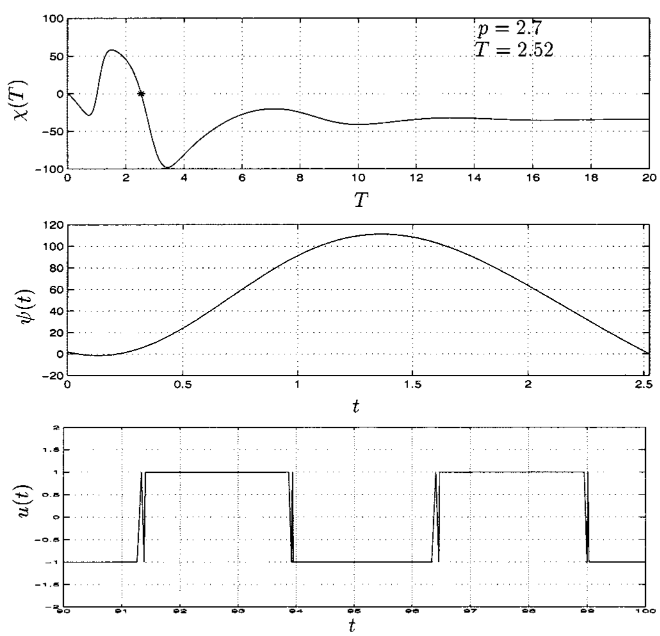


Fig. 7. Graph of $\chi(T)$ and $\psi(t)$ just after the bifurcation and $u(t)$ for the resulting orbit in *Example 5*.

III. Entrainment, Synchrony, and Phase Locking

We now consider the response of a relay relaxation oscillator under periodic external forcing. Forcing an autonomous oscillator with a periodic signal produces an interesting phenomenon known as *entrainment* or *resonance*. When the frequency of the forcing signal (Ω) is *close to* being a rational multiple (ρ) of the natural or autonomous frequency (ω_n) of the oscillator, it can be entrained, i.e., the system adapts itself and responds with a frequency $\omega = \rho\Omega$. A generalization of forcing is a system of coupled oscillators where each oscillator can influence some or all the others and *vice versa*. Coupled oscillator systems exhibit *synchrony* and *phase locking*. Synchrony is a state wherein all oscillators in the network oscillate with the same phase. In phase locking, each oscillator maintains a constant phase difference with respect to the others. This subject has been studied in great detail in many physical, chemical and biological systems (see [6]–[8], [27], [36], [37], and the references therein).

In the remainder of the paper, we focus on a similar analysis for the relay relaxation oscillator. We treat in detail the simple but representative case where the linear system consists of an integrator (see Fig. 9). We demonstrate entrainment at the primary harmonic ($\omega = \Omega$) and other subharmonics ($\omega = \Omega/m, m$ integer). We also study synchronization when two or three relay relaxation oscillators are coupled in a ring. The analysis can be extended to n-member rings.

Models of relaxation oscillators have a fast and a slow subsystem and hence, two possible ports for input and two for output. Thus, there are two qualitatively different kinds of forcing depending on the input port used. The mechanism for entrainment in piecewise linear models of relaxation oscillators described in [7] and the analysis of coupled relaxation oscillators in [28] and [38] consider input to the fast subsystem whereas in our analysis we consider input to the slow subsystem (the integrator). When relay hysteresis is considered, the analysis in [8] corresponds to taking the output of the slow subsystem, whereas we consider the output from the relay which is the fast subsystem. Furthermore, relay switchings can be imagined as *firings* and hence, coupled relay systems can be viewed abstractly as pulse coupled oscillators. Dror *et al.* [29] analyzed rings of pulse-coupled oscillators using phase response curves under the assumption that the firing of an oscillator affects only the immediate next firing of the next oscillator in the ring. However, in coupled relay oscillators, even in a situation close to entrainment, a switching can influence *two* future switchings of the next oscillator. This makes the phase response curve method unsuitable. Using the technique described in [27], the relay oscillator system with weak forcing input to the slow subsystem can be reduced to a piecewise continuous flow on the torus, and the entrainment behavior can be obtained by studying the underlying circle map. However, we use a different technique that works for the case of strong forcing also. We not only obtain the same results in the weak forcing regime as with the circle map technique, but also obtain the bistability regions under strong forcing in the overlapping region of primary and subharmonic entrainment.

In this paper, we restrict our attention to *regular* dynamics, i.e., periodic states but forced relay systems can exhibit complex chaotic dynamics under strong forcing due to the presence of *stretching and folding* action [6, p. 339]. However since the fast subsystem is not directly affected by the coupling, the mechanism for stretching and folding is through the acceleration and deceleration of the initial conditions moving along the branches of the relay.

A. Primary and Subharmonic Entrainment

Consider the configuration of Fig. 9 where a relay relaxation oscillator of natural half-period T_1 is being driven by an external periodic signal v. Notice the change of notation in Fig. 9 where a negative feedback is used at the integrator. The signal v is taken to be a square-wave of amplitude $\epsilon > 0$ and half-period T. For primary or subharmonic entrainment the particular shape of the signal is not important but only its $\mathcal{L}_1$-content over its half-period. This is because it does not matter how the output of the integrator $v(t)$ varies except when it reaches the switching points ± 1. Given any zero-mean forcing signal, it is always possible to find an equivalent square wave that preserves the switching times of the relay and hence $u(t)$. In the analysis below, we will identify the entrainment regions in the σ–ϵ parametric plane where $\sigma := T_1/T$ is the normalized autonomous period. The results of this section are summarized in Fig. 10. Only the primary entrainment is possible in the regions P_1 and P_2. In B_1 and B_2, the system exhibits bistability and both the primary and secondary subharmonic entrainment are possible. Higher subharmonics and multistability can be observed in the triangular regions emanating from the other odd integers on σ-axis which are not analyzed here.

Let t_k $(k = 1, 2, \ldots)$ denote the times at which the state of the relay switches between $+1$ and -1. We index the switching times with the convention $(k-1)T \le t_k < kT$ as shown in Fig. 9. We define as *entrainment* (at the fundamental or primary harmonic with period $2T$) the situation where

$$(k-1)T \le t_{\ell+k} < kT$$

for all k sufficiently large and some fixed value of $\ell \in \{0, 1, 2 \ldots\}$, i.e., after sufficiently long time the switchings of the forcing signal and relay alternate. Similarly, we often talk about entrainment in subharmonic resonances when the switching takes place at regular intervals missing an integral number of periods of the forcing signal. We say that the oscillator is *entrapped* if its initial state is such that entrainment

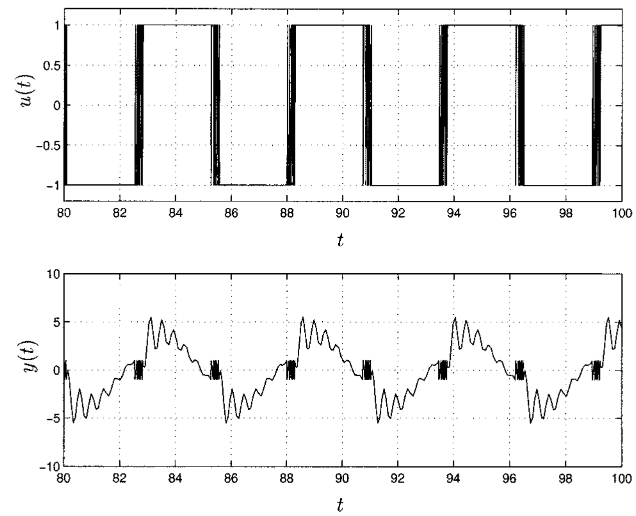


Fig. 8. $u(t)$ and $y(t)$ for the orbit corresponding to $p = 15$ in *Example 5*.

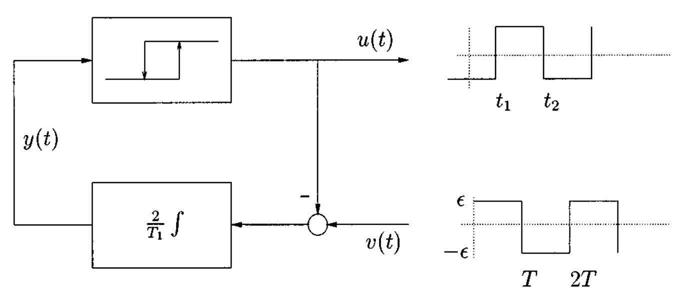


Fig. 9. Relay relaxation oscillator with a forcing input.

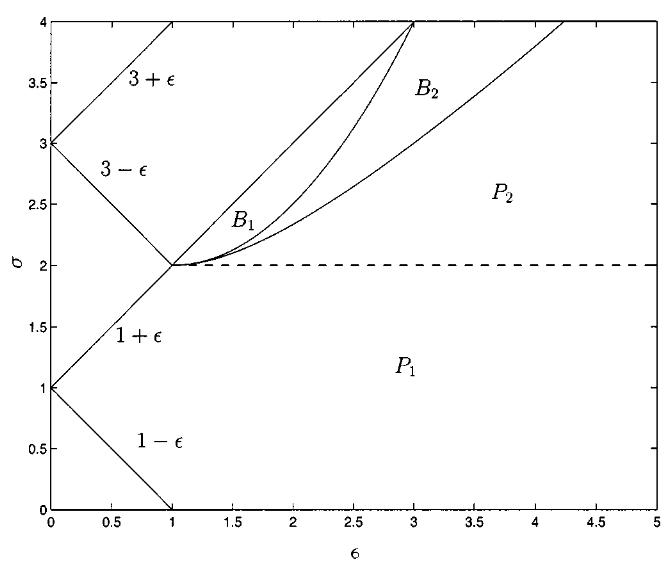


Fig. 10. Entrainment regions for the relay oscillator.

eventually takes place, i.e., if the initial phase lies in the domain of attraction of the entrained state.

The approach we use to prove entrainment involves assuming a certain switching pattern and then investigating the existence and stability of fixed points consistent with that pattern. It is convenient to work with a phase variable representing the fractional lead or lag of u w.r.t v in stead of the switching times. We define the *phase* at the kth switching as $\theta_k := t_k/T-(k-1)$. For the forced relay system, two switching patterns are possible *viz. in-phase* and *out-of-phase*. If the relay switches to ± 1 during $v=\pm\epsilon$ respectively, we call it in-phase switching and the phase is positive i.e., $0 \le \theta < 1$. Similarly if the relay switches to ± 1 occur during $v = \mp\epsilon$, it is out-of-phase switching with $-1 \le \theta < 0$. Fig. 9 shows the in-phase switching pattern whereas changing the sign of $u(t)$ corresponds to out-of-phase switching.

Proposition 5: For the oscillator of Fig. 9, driven by the square wave v of amplitude ϵ and half-period T, entrainment (at the fundamental harmonic) is possible iff

$$1-\epsilon \le \sigma < 1+\epsilon. \tag{42}$$

Subharmonic resonance with period $2(m+1)T$ is also possible in case

$$1-\epsilon+2m \le \sigma < 1+\epsilon+2m. \tag{43}$$

If $\sigma \le 2$, the oscillator is entrapped at the fundamental harmonic for almost all values of the initial phase $-1 \le \theta_1 < 1$. In general, intervals exist for the values of the initial phase where the oscillator is entrapped in subharmonic resonance.

Proof: 1) Necessary Conditions for Entrainment—Assume that the first switching from -1 to $+1$ takes place at t_1. We first consider the case of in-phase entrainment, $0 \le \theta_k < 1$ $(k = 1, 2, \ldots)$. Switching at two subsequent values for the phase θ_k and θ_{k+1} $(k = 1, 2, \ldots)$ requires that the state of the integrator changes under the influence of the driving and feedback signals by -2, giving

$$[(1-\theta_k)-\theta_{k+1}]\,\frac{2}{\sigma}\,\epsilon - [(1-\theta_k)+\theta_{k+1}]\,\frac{2}{\sigma} = -2.$$

Hence,

$$\theta_{k+1} = \beta + \alpha\theta_k \tag{44}$$

where $\alpha = (1-\epsilon/1+\epsilon)$ and $\beta = \sigma/(1+\epsilon)-\alpha$. Since $\epsilon > 0$, $|\alpha| < 1$, and (44) have a fixed point

$$\theta = \frac{\sigma-(1-\epsilon)}{2\epsilon}. \tag{45}$$

By assumption, the phase at steady stated lies in [0, 1) giving

$$1-\epsilon \le \sigma < 1+\epsilon. \tag{46}$$

Clearly, this condition is also sufficient since an initial phase $\theta_1 = \theta$ leads to entrainment. Since (44) has a Lipschitz constant α with $|\alpha| < 1$, there is at least an open interval about the fixed point where entrapment is guaranteed.

We now consider the case of out-of-phase entrainment, $-1 \le \theta_k < 0$ $(k = 1, 2, \ldots)$. Switching at two subsequent values for the phase θ_k and θ_{k+1} now leads to

$$[-(-\theta_k)+(1+\theta_{k+1})]\;\frac{2}{\sigma}\,\epsilon - [(-\theta_k)+(1+\theta_{k+1})]\;\frac{2}{\sigma} = -2$$

and, therefore, to

$$\theta_{k+1} = \frac{\sigma}{1-\epsilon} + \frac{1+\epsilon}{1-\epsilon}\,\theta_k - 1.$$

Since $\epsilon > 0$, then $|(1+\epsilon)/(1-\epsilon)| > 1$, ruling out the possibility of a stable fixed-point $-1 \le \theta < 0$. Thus, in an entrainment situation the sign of the oscillator necessarily follows the sign of the driving signal and $0 \le \theta_k < 1$.

2) Range of θ_1 for Entrapment at the Primary Harmonic—We consider separately the two cases $0 \le \theta_1 < 1$ and $-1 \le \theta_1 < 0$ for the initial phase.

Case (a): Let $0 \le \theta_1 < 1$. Switching cannot take place before $t = T$ since the state of the integrator at $t = T$ is $1 + (\epsilon-1)(1-\theta_1)(2/\sigma) > -1$ by virtue of the fact that $\sigma > 1-\epsilon$. Switching over the next half-period interval $[T, 2T]$ requires that $1+(\epsilon-1)(1-\theta_1)(2/\sigma)-(\epsilon+1)(2/\sigma) < -1$ which gives $(1-\epsilon)\theta_1 < 2-\sigma$. In case $\epsilon < 1$, this always holds since $\sigma < \epsilon+1$. In case $\epsilon > 1$, it still holds whenever $\sigma < 2$. The same holds true for all k and since $\theta_{k+1} = \alpha\theta_k + \beta$ with $|\alpha| < 1$, in both cases, entrainment takes place.

We now consider separately the case where $\sigma > 2$ (and, hence, $\epsilon > 1$). In this case, the possibility exists for switching to take place after the oscillator has missed an integral number of periods, $2T$, giving rise to *subharmonic resonance*. In particular, if the state of the integrator at $t = 2T$ is

$$1+(\epsilon-1)(1-\theta_1)\,\frac{2}{\sigma} - (\epsilon+1)\,\frac{2}{\sigma} > -1$$

then it is still greater than -1 at $3T$ and and it may only reach -1 over the next half period $[3T, 4T]$, at a phase θ such that $1+(\epsilon-1)(1-\theta_1)(2/\sigma)-(4/\sigma)-(\epsilon+1)\theta(2/\sigma) = -1$. This gives switching at $\theta_4 = \theta$ (with θ_2, θ_3 missing) where

$$\theta_4 = \alpha\theta_1 + \beta_1 \tag{47}$$

with $\beta_1 = (\sigma-2)/(1+\epsilon)-\alpha$. The iteration in (47) is still a (Lipschitz-) contractive map from θ_1 to θ_4 leading to a fixed point phase value of $\theta_{(1)} = (\beta_1/1-\alpha)$. A necessary condition for entrainment at this subharmonic resonance is that $0 \le \beta_1/(1-\alpha) < 1$ which yields

$$3-\epsilon \le \sigma < 3+\epsilon.$$

Similarly, in the case of

$$1+2m-\epsilon \le \sigma < 1+2m+\epsilon$$

other subharmonic resonances are possible where switching occurs after missing m integral periods $(m = 2, 3, \ldots)$ settling at $\theta_{(m)} = \beta_m/(1-\alpha)$ with $\beta_m = (\sigma-2m)/(1+\epsilon)-\alpha$. In all these cases intervals of entrapment at the subharmonic resonance exist, as well as the possibility of a more complicated behavior in general (provided σ, ϵ are sufficiently large).

We now identify intervals of entrapment at the primary harmonic. First note that, in general when $0 \leq \theta_k \leq 1$, switching cannot take place in the interval $[(k-1)T, kT]$ since $1 + (\epsilon - 1)(2/\sigma)(1-\theta_k) > -1$ by virtue of the fact that $\sigma > 1 - \epsilon$. In the case where $\sigma < 2$, the next switching will necessarily take place in $[kT, (k+1)T]$ since $1 + (\epsilon - 1)(2/\sigma)(1-\theta_k) - (\epsilon+1)(2/\sigma) < -1$ by virtue of the conditions $\sigma < \epsilon + 1$ and $\sigma < 2$ and entrapment takes place. We now deal with the case $\sigma > 2$. Since $\sigma < 1 + \epsilon$, then $\epsilon > 1$. Define

$$\delta := \frac{\sigma - 2}{\epsilon - 1}.$$

A simple computation shows that if $\delta \leq \theta_k \leq 1$, switching will take place in the next half-period, and $1 \geq \theta_{k+1} \geq (\sigma/1+\epsilon)$. Here, $\sigma < 1+\epsilon$ implies that $(\sigma/1+\epsilon) > \delta$ which in turn shows that $\theta_1 \in [\delta, 1]$ leads to entrainment at the fundamental harmonic. In general, there are other intervals in [0, 1] where entrapment at the fundamental, or other harmonics, is also possible. The structure of these intervals is complicated in general and will only be elucidated in Proposition 6 for the case $2 < \sigma < 4$.

Case (b): We finally consider the case $-1 < \theta_1 < 0$. We show that the phase in subsequent intervals drifts and eventually θ_k (for some k) lies in [0, 1]. Thereby the transition toward entrainment follows the previous analysis.

In case $\epsilon > 1$, no switching out-of-phase is possible in subsequent intervals since the external driving signal exceeds the value of the feedback. Eventual switching is unavoidable since the value of v reduces by $-4/\sigma$ over every subsequent $2T$-interval. Therefore, the next switching will take place in-phase, giving a positive value for the relevant θ_k, which brings us back to the previous analysis. Entrainment at the fundamental frequency or at another subharmonic resonance will depend on the particular value of θ_k in the first in-phase switching.

Now consider the case $\epsilon < 1$. We again show that in-phase operation will eventually take place, albeit, after possibly a large number of intervals. In case $-1 < \theta_1 < d_0$ with $d_0 := -\sigma/(\epsilon+1)$ the next switching takes place in the same interval $[-T, 0]$, since at a point $\theta < 0$ the value of y becomes -1, giving rise to in-phase operation (then just rename $\theta \to \theta_{-1}$). To see this note, that if no switching were to take place the value of $y(0)$ would be $1 + (-\epsilon - 1)(2/\sigma)(-\theta_1) < 1$. Define $d_k := d_0 + \alpha(1 + d_{k-1})$ for $k = 1, 2, \ldots$ and set $d_{-1} := -1$. It can be checked that if out-phase switching takes place with phase in $[d_k, d_{k+1}]$ ($k \geq 0$), the next switching will take place over the next half-interval period (hence still out-of-phase) with value for the corresponding phase in $[d_{k-1}, d_k]$. Hence, after a finite number of steps, the phase will be in the interval $[-1, d_0]$ and in the next step in-phase operation will be restored. It only remains to show that the union of the intervals $[d_{k-1}, d_k]$ for $k \geq 0$ covers $[-1, 0]$. This follows from the fact that $d_k \to (1+d_0/1-\alpha) > 0$ as k increases (since $\epsilon < 1 \Rightarrow 0 < \alpha < 1$ while $-1 < d_0 < 0$). ∎

We wish to elucidate a case where subharmonic resonance is possible. The general situation is quite complex, so we will present only the analysis of the case where $2 < \sigma < 4$, in which case the only subharmonic resonance possible is the one with period $4T$.

Proposition 6: Consider the oscillator of Fig. 9, driven by the square-wave v of amplitude ϵ and half-period T, and with normalized period σ satisfying $2 < \sigma < 4$.

1) If $2 < \sigma < 2 + (\epsilon-1)^2/(\epsilon+1)$, the oscillator is always entrapped at the fundamental harmonic.
2) If $2 + (\epsilon-1)^2/(\epsilon+1) < \sigma < 2 + (\epsilon-1)^2/2$, the oscillator is entrapped at the fundamental harmonic when
$$\theta_1 \in \left[0, 1 - \frac{2(\sigma-2)}{(\epsilon-1)^2}\right] \cup \left[\frac{\sigma-2}{\epsilon-1}, 1\right]$$
and in a subharmonic resonance with period $4T$ otherwise.
3) If $2 + (\epsilon-1)^2/2 < \sigma < 4$, the oscillator is entrapped at the fundamental harmonic when $\theta_1 \in [(\sigma-2)/(\epsilon-1), 1]$ and in a subharmonic resonance with period $4T$ otherwise.

Proof: We deal only with initial in-phase switching, i.e., $0 < \theta_1 < 1$. From the Proof of Proposition 5 we know that a value of θ_1 in the interval $[\delta, 1]$ with $\delta = (\sigma-2)/(\epsilon-1)$ leads always to entrapment at the fundamental harmonic. On the other hand, if $\theta_1 < \delta$ the oscillator skips two half-period intervals and switches again at

$$\theta_4 = \alpha\theta_1 + \frac{\sigma-2}{\epsilon+1} - \alpha. \tag{48}$$

The map in (48) takes the interval $[0, \delta]$ onto the interval $[\delta', o']$ where $\delta' := (\epsilon-1)/(\epsilon+1)$ and $o' := (\sigma+\epsilon+3)/(\epsilon+1)$. Another important quantity is the pre-image $\hat{\delta}$ of δ under the same map. This is $\hat{\delta} = 1 - 2(\sigma-2)/(\epsilon-1)^2)$.

There are three cases of interest: a) when under the action of the map in (48), $[\delta', o']$ lies squarely in the domain of attraction of the fundamental resonance; b) when it lies in the domain of attraction of the subharmonic resonance; and c) the *mixed* case. Thus, in case $\delta < \delta'$ all initial phases lead to entrainment at the fundamental harmonic, albeit after the interval $[T, 3T]$. This condition leads to the condition given in part 1) of Proposition 6. In case $o' < \delta$, all initial phases in the interval $[0, \delta]$ lead to phase values in the same interval after missing a complete $2T$ period. This gives rise to condition in part 3) of Proposition 6. The mixed case requires considering the pre-image of δ, $\hat{\delta}$, and its relative position with regard to δ'. It turns out that $\hat{\delta} < \delta'$ holds always. Therefore, all values of $\theta_1 < \hat{\delta}$ lead to entrapment at the fundamental resonance, while all values larger than $\hat{\delta}$ and less than δ lead to subharmonic resonance. This gives the conditions and the particular intervals in part 2) of *Proposition 6*. ∎

B. Synchrony and Phase Locking in Coupled Relay Oscillators

Rings of coupled relay relaxation oscillators can exhibit synchrony and phase-locking phenomena ([30], [31]) for suitable coupling strengths. Here, we demonstrate these phenomena for the two and three member rings shown in Fig. 11. A single oscillator unit with autonomous half-period T_a along with its input and output ports is shown in Fig. 11(a). The method used is analogous to the case of entrainment analysis in Section III-A. We enumerate all the switching patterns for the oscillators with the assumption that no oscillator switches twice before all the others in the ring switch, and investigate the existence of fixed points and their local stability. The above assumption implies that complex switching patterns are not considered and the situations analyzed cover only a part of the state space. Therefore, the results are only sufficient conditions for synchrony or phase locking.

The two oscillators O_1 and O_2 shown in Fig. 11(b) have autonomous half-periods T_1 and T_2 and are connected in a feedback loop with gains γ and $-\gamma$ respectively. Without loss of generality, assume that $T_2 \geq T_1$. We investigate the possibility of the feedback interconnection attaining *synchrony*, that is, both components oscillating with a common period. The conditions for synchrony are stated in the following proposition. Two oscillator ring with the same sign for the gains cannot attain synchrony for $T_1 \neq T_2$ and will not be discussed here.

Proposition 7: The two oscillator system in Fig. 11(b) with $T_2 \geq T_1$ attains synchrony with (a) O_2 leading O_1 for $\gamma > (T_2 - T_1)/(T_2 + T_1)$ and (b) with O_1 leading O_2 for $-\gamma > (T_2 - T_1)/(T_2 + T_1)$ with the mean half-period $T := (T_1 + T_2)/2$.

Proof: The two scenarios are illustrated in Fig. 12. Observe that if a gain γ produces the scenario (a), then the symmetry of the ring in Fig. 11(b) implies that a gain $-\gamma$ produces the scenario (b). Hence we need to prove only part (a).

Consider the scenario shown in Fig. 12(a). Let t_k^i be the k-th switching time of the i-th oscillator. Tracing the states of the integrators of the two oscillators, we obtain

$$T_1 = t_{k+1}^1 - t_k^1 - \gamma(t_k^2 - t_k^1) + \gamma(t_{k+1}^1 - t_k^2) \quad (49)$$
$$T_2 = t_{k+1}^2 - t_k^2 - \gamma(t_{k+1}^1 - t_k^2) + \gamma(t_{k+1}^2 - t_{k+1}^1). \quad (50)$$

Let the half-period at synchrony be T and $\theta^0 T$ be the lag of O_1 w.r.t O_2, i.e., $t_{k+1}^i - t_k^i = T$ and $t_{k+1}^1 - t_k^2 = \theta^0 T$ for all k. Substituting in (49), (50), we get $T = (T_1 + T_2)/2$ and $\theta^0 = 1/2 - (T_2 - T_1)/(4\gamma T)$. The constraints $0 \leq \theta^0 \leq 1$ give rise to $\gamma > (T_2 - T_1)/(T_2 + T_1)$.

Now, to analyze the stability of the synchronous state, define the phase variables $\theta_k^i := t_k^i - kT$ and denote $\theta_k = [\theta_k^1 \;\; \theta_k^2]'$. Substituting (49), (50), we get

$$\begin{bmatrix} 1+\gamma & 0 \\ 2\gamma & 1+\gamma \end{bmatrix} \theta_{k+1} = \begin{bmatrix} 1-\gamma & -2\gamma \\ 0 & 1-\gamma \end{bmatrix} \theta_k.$$

The eigenvalues of this system are 1 and $(1-\gamma)/(1+\gamma)^2$. The unity eigenvalue has the eigenvector $[1\ 1]'$ and corresponds to a constant shift in time. The second eigenvalue has modulus less than unity since $\gamma > 0$. Hence, the synchronous state shown in Fig. 12(a) is stable. ■

Remark 8: From a control perspective, *Proposition 7* implies that both the period and phase of a relay relaxation oscillator can be set to arbitrary values using an oscillatory feedback controller and proper choice of gains.

Now let us consider a ring of three coupled oscillators. For simplicity, we assume that the oscillators are identical. Fig. 11(c) and (d) shows the cases of one-way coupling and two-way coupling respectively. Again, a two-way coupled system with gains having the same sign in both directions does not show stable phase locking behavior and will not be considered. We can represent the switching scenarios using a compact

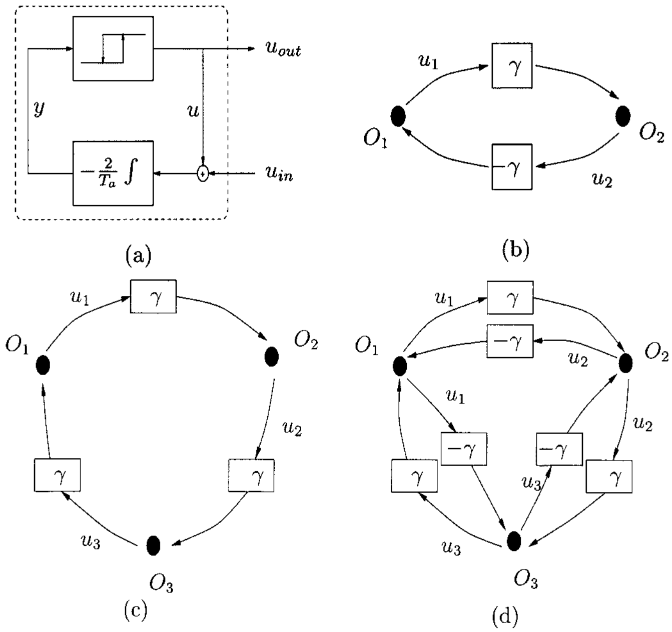


Fig. 11. Coupled relay oscillators. (a) single oscillator with input and output ports, (b) two member ring with gains $\pm\gamma$, (c) three member ring with one-way coupling, and (d) three member ring with two-way coupling.

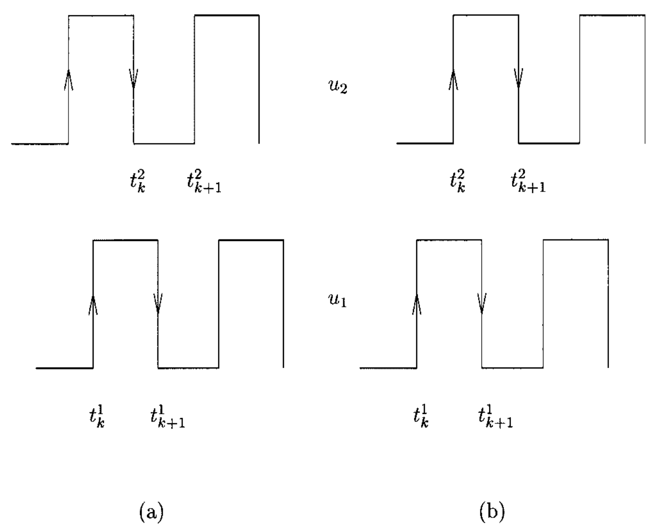


Fig. 12. Two scenarios for synchrony in the two oscillator system: (a) O_2 leads O_1, (b) O_1 leads O_2.

notation in stead of pictures. For example the two scenarios in Fig. 12 can be represented as $\{1 \uparrow 2 \downarrow \cdots\}$ and $\{1 \uparrow 2 \uparrow \cdots\}$ respectively. Of all the simple switching patterns (recall our assumption that no oscillator switches twice before the others switch) for the three member ring, only two lead to stable phase locking. The following propositions summarize the results for three member rings. Assume, without loss of generality, that the oscillators have unit autonomous half-period.

Proposition 8: Consider the three oscillator system with one-way coupling shown in Fig. 11(c) where all the oscillators are identical and have unit autonomous half-period. The system attains a 1/3 phase locked state with half-period $T = 3/(3-\gamma)$ and the switching pattern (a) $\{1 \uparrow 2 \downarrow 3 \uparrow \cdots\}$ for $0 < \gamma < 3$ and (b) $\{1 \uparrow 3 \downarrow 2 \uparrow \cdots\}$ for $-1 < \gamma < 0$.

Proof: Consider the switching pattern $\{1\uparrow 2\downarrow 3\uparrow \cdots\}$ and let t_k^i denote the kth switching time of the ith oscillator with the convention that $t_k^1 \le t_k^2 \le t_k^3$. Tracing each of the oscillators, we obtain

$$1 = t_{k+1}^1 - t_k^1 - \gamma(t_k^3 - t_k^1) + \gamma(t_{k+1}^1 - t_k^3) \quad (51)$$
$$1 = t_{k+1}^2 - t_k^2 - \gamma(t_{k+1}^1 - t_k^2) + \gamma(t_{k+1}^2 - t_{k+1}^1) \quad (52)$$
$$1 = t_{k+1}^3 - t_k^3 - \gamma(t_{k+2}^1 - t_k^3) + \gamma(t_{k+1}^3 - t_{k+1}^2). \quad (53)$$

Observe that there is a fixed point with a phase difference of 1/3. Let T be the resulting half-period, i.e., $t_{k+1}^i - t_k^i = T$ and set $t_k^2 - t_k^1 = t_k^3 - t_k^2 = T/3$ for all k. Substituting in (51), we obtain $T = (3/3 - \gamma)$.

To analyze the stability, define the phases $\theta_k^i := t_k^i - kT - (i-1)T/3$ and let $\theta_k = [\theta_k^1 \;\; \theta_k^2 \;\; \theta_k^3]'$. Substituting (51)–(53), we get

$$\begin{bmatrix} 1+\gamma & 0 & 0 \\ -2\gamma & 1+\gamma & 0 \\ 0 & -2\gamma & 1+\gamma \end{bmatrix} \theta_{k+1} = \begin{bmatrix} 1-\gamma & 0 & 2\gamma \\ 0 & 1-\gamma & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \theta_k.$$

The eigenvalues of this system are obtained using *Mathematica*® as 1, $[(\gamma-1)^2(1+2\gamma) \pm \gamma(\gamma-1)^{3/2}(5\gamma+3)^{1/2}]/(1+\gamma)^3$. The unity eigenvalue has the eigenvector $[1\ 1\ 1]'$ and corresponds to a constant shift in time. The nontrivial eigenvalues have modulus less than unity for $0 < \gamma < 3$. Hence, the synchronous state shown in Fig. 12(a) is stable for $0 < \gamma < 3$.

Similarly, considering the pattern $\{1\uparrow 3\downarrow 2\uparrow \cdots\}$ we can show that there is a 1/3 phase locked state with half-period $T = 3/(3-\gamma)$ and it is stable for $-1 < \gamma < 0$. ■

Along the same lines, we can analyze the two-way coupled three oscillator ring. We skip the proof and state the result. Notice that part (a) of the below proposition implies part (b) by symmetry.

Proposition 9: Consider the three oscillator system with two-way coupling shown in Fig. 11(d) where all the oscillators are identical and have unit autonomous half-period. The system attains a 1/3 phase locked state with half-period $T = 1$ and the switching pattern (a) $\{1\uparrow 2\downarrow 3\uparrow \cdots\}$ for $\gamma > 0$ and (b) $\{1\uparrow 3\downarrow 2\uparrow \cdots\}$ for $\gamma < 0$.

Remark 9: Observe that in the one-way coupled case, even excitatory coupling (positive γ) between the oscillators results in a state with out-of-phase locking where each oscillator responds in the opposite direction than the excitation signal.

A completely synchronous state with simultaneous switchings of all the three oscillators has been observed in the two-way coupled system in numerical simulations. But the approach to this state always violated the simple switching pattern assumption implying that the trajectory of the system goes out of the set of initial conditions we considered in this paper. The technique we have used can be extended to n-member rings to identify and analyze the stability of phase-locked states.

IV. Conclusion

A basic model for relaxation oscillators is that of a hysteresis in feedback with negative-integral action. The classical van der Pol oscillator can be viewed in this form. This structure has been identified in many chemical and biological systems and has also been used to design many chemical oscillators. This structure can be exploited to alter the behavior of an existing oscillatory system, e.g. by changing the integral action with a suitable outer loop. While there is an extensive literature on the subject of oscillations, the problem of predicting oscillations in higher dimensional systems, in general, is a difficult one.

In this paper, we have studied in detail unimodal periodic oscillations for the case of a feedback connection with a linear system and a relay hysteresis. Precise conditions for the existence and local stability are provided using the Poincaré map. The existence of a degenerate case with a continuum of periodic orbits is shown. Asymmetric orbits and orbits with many relay switchings per period have also been addressed. The methodology extends to the case of nonlinear systems connected to static hysteresis as well.

Relay oscillators can exhibit entrainment phenomena very much like other smooth oscillators, and can be used in coupled oscillator models. The particular case of pure integrator in feedback with relay hysteresis is similar to the van der Pol oscillator and is amenable to a detailed analysis. We study in detail primary and subharmonic entrainment in this system when forced by a periodic driving signal at the input to the integrator. We also consider the synchrony and phase locking in rings of two and three coupled oscillators. The technique can be applied to n member rings as well.

While oscillations are omnipresent in natural as well as in engineering systems, key questions still remain. In particular, identifying periodic orbits for general systems, e.g., in the optimal periodic control problem [39], [40], is quite challenging. On the other hand, robust control of periodic orbits is still a relatively unexplored area [41]. We expect that the paradigm of hysteresis with a linear feedback component will provide a useful framework for the further development of the subject.

References

[1] A. T. Winfree, *The Geometry of Biological Time*. New York: Springer-Verlag, 1980.

[2] J. Grasman and M. J. W. Jansen, "Mutually synchronized relaxation oscillators as prototypes of oscillating systems in biology," *J. Math. Biol.*, vol. 7, no. 2, pp. 171–197, 1979.

[3] I. R. Epstein and K. Showalter, "Nonlinear chemical dynamics: Oscillations, patterns and chaos," *J. Phys. Chem.*, vol. 100, no. 31, pp. 13131–13 147, 1996.

[4] A. Yu. Kolesov and Yu. S. Kolesov, "Relaxation oscillations in mathematical models of ecology," in *Proc. Steklov Inst. Math.*, vol. 199, 1995.

[5] J. Guckenheimer, "Dynamics of the Van der Pol equation," *IEEE Trans. Circuits Syst.*, vol. 27, pp. 983–989, Nov. 1980.

[6] E. A. Jackson, *Perspectives of Nonlinear Dynamics*. New York: Cambridge Univ. Press, 1991, vol. 1–2.

[7] M. Levi, "Qualitative analysis of the periodically forced relaxation oscillator," *Mem. Am. Math. Soc.*, vol. 32, no. 244, 1981.

[8] J. Grasman, "Asymptotic methods for relaxation oscillations and applications," in *Applied Mathematical Sciences*. New York: Springer-Verlag, 1987, vol. 63.

[9] S. K. Scott, *Oscillations, Waves and Chaos in Chemical Kinetics*. New York: Oxford Univ. Press, 1994.

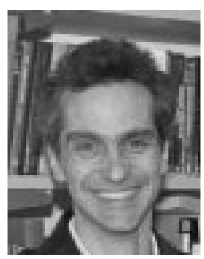
[10] M. A. Krasnosel'skiǐ and A. V. Pokrovskiǐ, *Systems with Hysteresis*. New York: Springer-Verlag, 1989.

[11] Ya. Z. Tsypkin, *Relay Control Systems*. New York: Cambridge Univ. Press, 1984.

[12] K. J. Aström *et al.*, "Recent advances in relay feedback methods-a survey," in *Proc. IEEE Conf. Systems, Man and Cybernetics*, vol. 3, 1995, pp. 2616–2621.

[13] K. J. Aström, "Oscillations in systems with relay feedback," in *Adaptive Control, Filtering and Signal Processing*, K. J. Aström, G. C. Goodwin, and P. R. Kumar, Eds. New York: Springer-Verlag, 1995, pp. 1–25.
[14] A. Megretski, "Global stability of oscillations induced by a relay feedback," in *Proceedings 13th IFAC World Congress*. New York: Pergamon, 1997, vol. E, pp. 49–54.
[15] C.-C. Yu, *Autotuning of PID Controllers: Relay Feedback Approach*. New York: Springer-Verlag, 1999.
[16] X. Zou and R. A. Siegel, "Modeling of oscillatory dynamics of a simple enzyme-diffusion system with hysteresis the case of lumped permeabilities," *J. Chem. Phys.*, vol. 110, no. 4, pp. 2267–2279, Jan. 1999.
[17] R. B. Northrop, *Endogenous and Exogenous Regulation and Control of Physiological Systems*. London: Chapman & Hall, U. K. 2000.
[18] J. W. Macki, P. Nistri, and P. Zecca, "Mathematical models for hysteresis," *SIAM Rev.*, vol. 35, no. 1, pp. 94–123, 1993.
[19] P. A. Bliman, A. M. Krasnosel'skii, M. Sorine, and A. A. Vladimirov, "Nonlinear resonance in systems with hysteresis," *Nonlinear Analy., Theory, Methods, & Appl.*, vol. 27, no. 5, pp. 561–577, 1996.
[20] M. Brokate and A. V. Pokrovskiĭ, "Asymptotically stable oscillations in systems with hysteresis nonlinearities," *J. Diff. Eq.*, vol. 150, no. 1, pp. 98–123, 1998.
[21] G. A. Leonov, I. M. Burkin, and A. I. Shepeljavyi, "Frequency methods in oscillation theory," in *Mathematics and Its Applications*. Norwell, MA: Kluwer, 1996.
[22] A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of Oscillators*. New York: Pergamon, 1966.
[23] Y. S. Kolesov, "Periodic solutions of n-th order equations with relay control," *Automat. Remote Contr.*, no. 11, pp. 1872–1875, 1969.
[24] G. Gripenberg, "On periodic solutions of a thermostat equation," *SIAM J. Math. Anal.*, vol. 18, no. 3, pp. 694–702, 1987.
[25] P. A. Bliman and A. M. Krasnosel'skiĭ, "Periodic solutions of linear systems coupled with relay," *Nonlinear Analy., Theory, Methods & Appl.*, vol. 30, no. 2, pp. 687–696, 1997.
[26] K. H. Johansson, A. Rantzer, and K. J. Aström, "Fast switches in relay feedback systems," *Automatica*, vol. 35, no. 4, pp. 539–552, 1999.
[27] G. DeYoung and H. G. Othmer, "Resonance in oscillatory and excitable systems," *Ann. New York Acad. Sci.*, vol. 591, pp. 128–153, June 1990.
[28] D. Somers and N. Kopell, "Rapid synchronization through fast threshold modulation," *Biol. Cybern.*, vol. 68, no. 5, pp. 393–407, 1993.
[29] R. O. Dror *et al.*, "A mathematical criterion based on phase response curves for stability in a ring of coupled oscillators," *Biol. Cybern.*, vol. 80, no. 1, pp. 11–23, 1999.
[30] J. J. Collins and I. Stewart, "Coupled nonlinear oscillators and the symmetries of animal gaits," *J. Nonlinear Sci.*, vol. 3, no. 3, pp. 349–392, 1993.
[31] S. H. Strogatz and I. Stewart, "Coupled oscillators and biological synchronization," *Sci. Amer.*, pp. 102–109, Dec. 1993.
[32] H.-A. Tanaka, A. J. Lichtenberg, and S. Oishi, "Self-synchronization of coupled oscillators with hysteretic responses," *Phys. D*, vol. 100, no. 3–4, pp. 279–300, 1997.
[33] R. F. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1977, vol. 1.
[34] J. M. Goncalves, A. Megretski, and M. A. Daleh, "Global stability of relay feedback systems," *IEEE Trans. Automat. Contr.*, 2000, to be published.
[35] M. di Bernardo, K. H. Johansson, and F. Vasca, "Sliding orbits and their bifurcations in relay feedback systems," in *Proc.IEEE CDC*, Phoenix, AZ, 1999.
[36] T. Kai and K. Tomita, "Stroboscopic phase portrait of a forced nonlinear oscillator," *Prog. Theor. Phys.*, vol. 61, no. 1, pp. 54–73, 1979.
[37] G. B. Ermentrout, "n:m phase-locking of weakly coupled oscillators," *J. Math. Biol.*, vol. 12, no. 3, pp. 327–342, 1981.
[38] N. Kopell and D. Somers, "Anti-phase solutions in relaxation oscillators coupled through excitatory interactions," *J. Math. Biol.*, vol. 33, no. 3, pp. 261–280, 1995.
[39] J. L. Speyer and R. T. Evans, "A second variational theory for optimal periodic processes," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 138–148, Feb 1984.
[40] S. Bittanti and G. Guardabassi, "Optimal periodic control and periodic systems analysis: An overview," in *Proc. 25th Conf. Decision Control*, Athens, Greece, Dec. 1986, pp. 1417–1423.
[41] A. L. Fradkov and A. Y. Pogromsky, *Intro. Contr. Oscillations Chaos*, Singapore: World Scientific, 1998, vol. 35 of A.

Subbarao Varigonda was born in Eluru, India, in 1975. He received the Bachelors degree in chemical engineering from the Indian Institute of Technology, Kanpur, India in 1996 and the Masters degree in electrical engineering in 1999 from the University of Minnesota, Minneapolis, where he is currently working toward the Ph.D. degree in chemical engineering.



Tryphon T. Georgiou (F'00) was born in Athens, Greece, in 1956. He received the Diploma in mechanical and electrical engineering from the National Technical University of Athens, Athens, Greece, and the Ph.D. degree from the University of Florida, Gainesville, in 1979 and1983, respectively.

He was a member of the faculty of Florida Atlantic University, Boca Raton, from 1983 to 1986, and of Iowa State University, Ames, from 1986 to 1989. Since August 1989, he has been with the University of Minnesota, where he is currently a Professor of Electrical and Computer Engineering. He has been an Associate Editor of the *SIAM Journal on Control and Optimization* and *Systems and Control Letters*.

Dr. Georgiou was a corecipient, together with Dr. Malcolm C. Smith, of the George Axelby Outstanding Paper award in 1992 and 1999. He has been an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL.