A Generalized Iterative Water-filling Algorithm for Distributed Power Control in the Presence of a Jammer

Ramy H. Gohary*, Yao Huang*, Zhi-Quan Luo* and Jong-Shi Pang†

* Department of Electrical and Computer Engineering

University of Minnesota, Minneapolis, MN 55455

† Department of Industrial and Enterprise Systems Engineering

University of Illinois, Urbana Champaign, IL 61801

Abstract

Consider a scenario in which K users and a jammer share a common spectrum of N orthogonal tones. Both the users and the jammer have limited power budgets. The goal of each user is to allocate its power across the N tones in such a way that maximizes the total sum rate that he/she can achieve, while treating the interference of other users and the jammer's signal as additive Gaussian noise. The jammer, on the other hand, wishes to allocate its power in such a way that minimizes the utility of the whole system; that being the total sum of the rates communicated over the network. For this non-cooperative game, we propose a generalized version of the existing iterative water-filling algorithm whereby the users and the jammer update their power allocations in a greedy manner. We study the existence of a Nash equilibrium of this non-cooperative game as well as conditions under which the generalized iterative water-filling algorithm converges to a Nash equilibrium of the game. The conditions that we derive in this paper depend only on the system parameters, and hence can be checked *a priori*. Simulations show that when the convergence conditions are violated, the presence of a jammer can cause the, otherwise convergent, iterative water-filling algorithm to oscillate.

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I. Introduction

The wireless communications spectrum is a scarce and valuable resource that is currently underutilized due to the usage of conventional static tone-assignment policies. This inherent drawback has been a fundamental reason behind the emergence of unlicensed open-spectrum communication systems [1], [2]. In these systems the spectrum is typically partitioned into N narrowband orthogonal tones and all users are allowed to access all the tones simultaneously and freely. In comparison with the fixed tone-assignment policies, this setup offers significantly greater freedom in utilizing the spectrum. However, this freedom comes at the expense of a number of challenges that ought to be taken into consideration by the system designer. In particular, the inherent overlap of the users' spectra in these systems gives rise to the so-called multi-user interference, which is a key limiting factor of open-spectrum communications. To mitigate the effect of multi-user interference, the users may employ a distributed power allocation mechanism whereby each user measures the interference level on each tone [3] and allocates its power dynamically across tones in such a way that maximizes its own utility.

With the increasing popularity of open-spectrum communication systems, it is conceivable that these systems will play an important role in future military communications. However, a major concern for these communications is the reliability with which the data can be transferred. For instance, open-spectrum communications may be susceptible to antagonistic behaviour of potential jammers that may be interested in reducing the utility of the entire system.[1] A jammer may be able to 'listen' to the users' transmissions, and to subsequently update its power allocation across tones in order to reduce the total sum-rate communicated over the network. As such, the procedure of both the users and the jammer can be cast as a non-cooperative game [4] in which the players are interested in maximizing their individual utilities in a selfish fashion.

In addition to open-spectrum communications, non-cooperative games arise in Digital Subscriber Line (DSL) systems in which the users compete to maximize their own utilities. For instance, in (jammer-free) DSL systems, the users may use the iterative water-filling algorithm (IWFA) [5] to allocate their powers across tones in such a way that maximizes their individual data rates. Being decentralized and relatively easy to implement, IWFA and variants thereof have been extended to scenarios in which the users may collaborate to maximize a common utility [6]–[8]. In order to gain insight into the inherent features of IWFA, several studies have focused on its convergence behaviour in both synchronous [9],

[1] In this paper, the sum rate of each user across tones will be referred to as the utility of the user, and the sum of utilities of all users will be referred to as the system utility.

[10] and asynchronous [11] scenarios. It is worth mentioning that while in IWFA the users compete to maximize their rate utilities, in other decentralized strategies the users may compete to maximize alternate jammer-free communication utilities; see e.g., [12]–[15]. In addition to jammer-free communication scenarios, the impact of malicious jamming has been considered in several studies. For instance, single-user systems in which the jammer's goal is to minimize the mutual information of the 'legitimate' user were considered in [16], [17], whereas multi-user single-tone communication systems in which the users' utilities are not directly related to rate utilities were considered in [18], [19].

Unlike these scenarios, in this paper we consider a communication system in which K users and a jammer share N orthogonal tones. Both the users and the jammer have limited power budgets. The goal of each user is to allocate its power across the N tones in such a way that maximizes the total sum rate that he/she can reliably communicate. The jammer, on the other hand, wishes to allocate its power in such a way that minimizes the utility of the whole system; that being the total sum of the rates communicated over the network. This scenario is analogous to a (zero-sum) non-cooperative game. In this paper we show that at least one Nash equilibrium exists for this game. Moreover, we develop a generalized version of the iterative water-filling algorithm (GIWFA) whereby users and the jammer update their power allocations in a greedy manner in order to maximize their respective utilities. The users and the jammer may update their power loads sequentially according to some prescribed order or they may update these loads in a totally asynchronous fashion at arbitrary time instants and using possibly outdated information about the interference from other users. We derive sufficient conditions under which GIWFA converges to a unique Nash equilibrium of this non-cooperative game, and we present numerical results that illustrate the impact of the jammer on the system utility and on the convergence of the users' iterates. In particular, we show that the presence of a strong jammer can not only reduce the total utility of the system, but also cause the otherwise convergent IWFA algorithm to oscillate.

The paper is organized as follows. Section II provides the system model, problem formulation, and the necessary definitions that will be used in subsequent sections. Section III contains the main results of the paper, including Nash equilibrium existence results and sufficient conditions for uniqueness. In Section IV we present some numerical experiments, and in Section V we provide some concluding remarks. For clarity of exposition, most of our proofs are relegated to the appendices.

II. System Model and Definitions

Consider a communication system in which N tones are shared by K user pairs and one jammer. In this paper we refer to a transmitter-receiver pair as one user, and we consider the case in which each user has one transmit and one receive antenna. Let h_{jk}^n denote the gain between the transmitter of User j and the receiver of User k at the n-th tones, for $j,k \in \mathcal{K}$ and $n \in \mathcal{N}$, where $\mathcal{K} \triangleq \{1,\ldots,K\}$ and $\mathcal{N} \triangleq \{1,\ldots,N\}$. Furthermore, let s_k^n and s_0^n be the power allocated by User k and the jammer to the n-th tone, respectively. Throughout this paper, the jammer will be denoted as User 0. If both the users and the jammer transmit Gaussian signals, then the rate that can be achieved by User $k \in \mathcal{K}$ on the n-th tone is given by [20]

$$R_k^n(s_1^n,\ldots,s_K^n) = \log\Big(1 + \frac{|h_{kk}^n|^2 s_k^n}{N_k^n + \sum_{j\neq k}|h_{jk}^n|^2 s_j^n + |h_{0k}^n|^2 s_0^n}\Big), \tag{1}$$

where N_k^n denotes the noise variance observed by User k on the n-th tone. By dividing both the numerator and the denominator by $|h_{kk}^n|^2$, the achievable rate of User $k \in \mathcal{K}$ on the n-th tone can be expressed as

$$R_k^n(s_1^n,\ldots,s_K^n) = \log\Big(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j\neq k}\alpha_{jk}^n s_j^n + \alpha_{0k}^n s_0^n}\Big), \tag{2}$$

where we define $\alpha_{0k}^n = |h_{0k}^n|^2/|h_{kk}^n|^2 \geq 0$, $\alpha_{jk}^n = |h_{jk}^n|^2/|h_{kk}^n|^2 \geq 0$, and $\sigma_k^n = N_k^n/|h_{kk}^n|^2 > 0$, for $j,k \in \mathcal{K}$, $n \in \mathcal{N}$. Suppose that User $k \in \mathcal{K}$, $(k \neq 0)$ is interested in maximizing its own sum-rate, so its utility is given by

$$U_k(\mathbf{s}_0,\mathbf{s}_1,\cdots,\mathbf{s}_K) = \sum_{n=1}^{N} R_k^n(s_1^n,\ldots,s_K^n) = \sum_{n=1}^{N}\log\Big(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j\neq k}\alpha_{jk}^n s_j^n + \alpha_{0k}^n s_0^n}\Big), \tag{3}$$

while the utility of the jammer is

$$U_0(\mathbf{s}_0,\mathbf{s}_1,\cdots,\mathbf{s}_K) = -\sum_{k=1}^{K} U_k = -\sum_{k=1}^{K}\sum_{n=1}^{N}\log\Big(1 + \frac{s_k^n}{\sigma_k^n + \sum_{j\neq k}\alpha_{jk}^n s_j^n + \alpha_{0k}^n s_0^n}\Big), \tag{4}$$

where we use $\mathbf{s}_k$ to denote the vector $[s_k^1,\cdots,s_k^N]^T$.

Given a limited power budget, and a maximum power constraint on each tone, the goal of User k, is to maximize U_k; that is, User k wishes to solve the following optimization problem,

$$\begin{aligned} \max \quad & U_k(\mathbf{s}_0,\mathbf{s}_1,\cdots,\mathbf{s}_K), \\ \text{subject to} \quad & \sum_{n=1}^{N} s_k^n \leq P_k, && \text{(5a)} \\ & 0 \leq s_k^n \leq S_{\max,k}^n, && \text{(5b)} \end{aligned}$$

where, P_k denotes the total power budget of User k, $S_{\max,k}^n$ denotes the maximum signal power that User k can use on the n-th tone, and in order for (5a) not to be redundant, we assume that $P_k \leq \sum_{n=1}^{N} S_{\max,k}^n$. We will denote the feasible set of User k as $\mathcal{P}_k$; that is,

$$\mathcal{P}_k \triangleq \big\{\mathbf{s}_k = [s_k^1,\cdots,s_k^N]^T \,|\, \sum_{n=1}^{N} s_k^n \leq P_k, 0 \leq s_k^n \leq S_{\max,k}^n\big\}. \tag{6}$$

Since individual users do not collaborate among themselves nor do they collaborate with the jammer, and both users and the jammer selfishly maximizes their own utilities, this communication scenario can be modelled as a non-cooperative game [4]. In this game individual users and the jammer are non-cooperative players, and the power allocations of any User k, including the jammer, that lie in $\mathcal{P}_k$ (cf., (6)) represent the set of admissible strategies of this user. A Nash equilibrium of this game [4] is a tuple of power strategies $\{\mathbf{s}_k^*\}_{k=0}^K$, such that for any $k \in \{0\} \cup \mathcal{K}$

$$U_k(\mathbf{s}_0^*, \mathbf{s}_1^*, \cdots, \mathbf{s}_{k-1}^*, \mathbf{s}_k^*, \mathbf{s}_{k+1}^*, \cdots, \mathbf{s}_K^*) \geq U_k(\mathbf{s}_0^*, \mathbf{s}_1^*, \cdots, \mathbf{s}_{k-1}^*, \mathbf{s}_k, \mathbf{s}_{k+1}^*, \cdots, \mathbf{s}_K^*), \quad \forall \mathbf{s}_k \in \mathcal{P}_k. \tag{7}$$

In other words, a Nash equilibrium of the game is a locally optimal strategy for each player that no player has an incentive to unilaterally change [4]. In the next section, we will show that, for this game, a Nash equilibrium always exists. Moreover, we will propose a decentralized algorithm for updating the jammer and the users' power allocations. By analyzing the convergence of this algorithm, we also derive sufficient conditions under which the Nash equilibrium is unique.

III. Existence and Uniqueness of a Nash Equilibrium

Since, for every $k = 1, \cdots, K$, $U_k(s_0, s_1, \cdots, s_{k-1}, \bullet, s_{k+1}, s_K)$ is a continuously differentiable concave function, and so is $U_0(\bullet, s_1, \cdots, s_K)$, and since each $\mathcal{P}_k$ is a compact convex set, it follows readily from [21, Proposition 2.2.9] that a Nash equilibrium exists. Such an equilibrium can be found using a standard fixed-point algorithm, an instance of which is given in the next section.

A. A generalized iterative water-filling algorithm (GIWFA)—Synchronous Version

In the jammer-free case, it can be shown that a certain transformation [10] can be invoked to expose an inherent equivalence between standard IWFA and the fixed-point algorithm [21], [22]. Drawing on this observation, we devise a generalized water-filling algorithm (GIWFA) whereby the users and the jammer update their power allocations using fixed-point iterations. In particular, let $s_k^{n,\nu}$ be the power allocation of User k on the n-th tone at iteration ν, and $\mathbf{s}_k^\nu$ be the vector $[s_k^{1,\nu}, \cdots, s_k^{N,\nu}]^T$. For the time being consider synchronous operation, whereby the users update their power allocations sequentially. Assume, without loss of generality, that the users are ordered so that User 1 updates its power allocation first then User 2 and so on, and that the jammer (User 0) updates its power allocation last. Hence, in each iteration User $k \in \mathcal{K}$ updates its power allocations in order to solve

$$\mathbf{s}_k^{\nu+1} = \left[\mathbf{s}_k^{\nu+1} + \nabla_{\mathbf{s}_k} U_k(\mathbf{s}_0^\nu, \mathbf{s}_1^{\nu+1}, \cdots, \mathbf{s}_{k-1}^{\nu+1}, \mathbf{s}_k, \mathbf{s}_{k+1}^\nu, \cdots, \mathbf{s}_K^\nu)\Big|_{\mathbf{s}_k = \mathbf{s}_k^{\nu+1}}\right]_{\mathcal{P}_k}, \tag{8}$$

whereas the jammer solves

$$\mathbf{s}_0^{\nu+1} = \left[\mathbf{s}_0^{\nu+1} + \nabla_{\mathbf{s}_0} U_0(\mathbf{s}_0, \mathbf{s}_1^{\nu+1}, \cdots, \mathbf{s}_K^{\nu+1})\Big|_{\mathbf{s}_0=\mathbf{s}_0^{\nu+1}}\right]_{\mathcal{P}_0}, \tag{9}$$

where we use $[\cdot]_{\mathcal{P}_k}$ to denote the projection operator onto the polyhedron defined in (6). That is, for any vector $x \in \mathbb{R}^N$

$$[x]_{\mathcal{P}_k} = \arg\min_{y \in \mathcal{P}_k} \|y - x\|. \tag{10}$$

Using (3) and (4), we can compute the gradients $\nabla_{\mathbf{s}_k} U_k$ explicitly. In particular, the n-th entry of $\nabla_{\mathbf{s}_k} U_k$ for $k \in \{0\} \cup \mathcal{K}$, $[\nabla_{\mathbf{s}_k} U_k]_n$, can be expressed as

$$\left[\nabla_{\mathbf{s}_k} U_k(\mathbf{s}_0^{\nu}, \mathbf{s}_1^{\nu+1}, \cdots, \mathbf{s}_{k-1}^{\nu+1}, \mathbf{s}_k, \mathbf{s}_{k+1}^{\nu}, \cdots, \mathbf{s}_K^{\nu})\Big|_{\mathbf{s}_k=\mathbf{s}_k^{\nu+1}}\right]_n = \frac{1}{\sigma_k^n + \sum_{j=1}^{k} \alpha_{jk}^n s_j^{n,\nu+1} + \sum_{j=k+1}^{K} \alpha_{jk}^n s_j^{n,\nu} + \alpha_{0k}^n s_0^{n,\nu}}, \qquad \forall k \in \mathcal{K}, \tag{11}$$

$$\left[\nabla_{\mathbf{s}_0} U_0(\mathbf{s}_0, \mathbf{s}_1^{\nu+1}, \cdots, \mathbf{s}_K^{\nu+1})\Big|_{\mathbf{s}_0=\mathbf{s}_0^{\nu+1}}\right]_n = \sum_{k=1}^{K} \frac{\alpha_{0k}^n s_k^{n,\nu+1}}{(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n s_k^{n,\nu+1} + \sigma_k^n + \alpha_{0k}^n s_0^{n,\nu})(\sum_{j=1}^{K} \alpha_{jk}^n s_k^{n,\nu+1} + \sigma_k^n + \alpha_{0k}^n s_0^{n,\nu})}, \tag{12}$$

where, in (11) and (12), we have used that $\alpha_{kk}^n = 1$ for all $k \in \mathcal{K}$.

From (11) and (12) we observe that for User $k \in \mathcal{K}$ to update its power allocation, it is sufficient to measure its own noise-plus-interference level on each tone, whereas for the jammer to update its power allocation, it needs, not only to know the power transmitted by each user, but also to know the noise-plus-interference level experienced by each user on every tone. One way for the jammer to acquire this information is to use standard means to estimate the physical locations of users. Using these locations and the tone frequencies, relatively accurate estimates of the (absolute) channel gains can be obtained using (empirical) frequency-dependent path-loss formulae for various propagation environments [23, Chapter 2]. (Channel phase information is not required for GIWFA.) Finally, by estimating the users' transmitted powers, the jammer can use the channel gain estimates to synthesize the interference patterns observed by the users.

It is worth noting that in the situations in which the jammer does not have full knowledge about the interference patterns observed by the users, the scenario considered in this work can be considered as a worst case scenario. Indeed, the jammer's impact on the system utility is more severe when it has full access to the interference patterns than if it has partial access only. This is because by having full access to the interference patterns, the jammer can determine the power allocations that minimize the overall system utility at each iteration of the algorithm.

B. Convergence Analysis—Synchronous Version

We now present sufficient conditions under which this algorithm converges to the unique Nash equilibrium of the game. Applying [22, Proposition 11.13] it can be seen that a tuple of power strategies $\{\mathbf{s}_k^*\}_{k=0}^K$ achieves equilibrium if and only if

$$\mathbf{s}_k^* = \left[\mathbf{s}_k^* + \theta \nabla_{\mathbf{s}_k} U_k(\mathbf{s}_0^*, \mathbf{s}_1^*, \cdots, \mathbf{s}_{k-1}^*, \mathbf{s}_k, \mathbf{s}_{k+1}^*, \cdots, \mathbf{s}_K^*)\Big|_{\mathbf{s}_k=\mathbf{s}_k^*}\right]_{\mathcal{P}_k}, \qquad k \in \mathcal{K} \tag{13a}$$

$$\mathbf{s}_0^* = \left[\mathbf{s}_0^* + \theta \nabla_{\mathbf{s}_0} U_0(\mathbf{s}_0, \mathbf{s}_1^*, \cdots, \mathbf{s}_K^*)\Big|_{\mathbf{s}_0=\mathbf{s}_0^*}\right]_{\mathcal{P}_0}, \tag{13b}$$

for some $\theta > 0$. Since our generalized iterative water-filling algorithm (8)–(9) corresponds to setting $\theta = 1$ in (13), then if this algorithm converges to a power strategy $\{\mathbf{s}_k^*\}_{k=0}^K$, then it must be a Nash equilibrium of the game (7). We now present sufficient conditions under which the generalized IWFA converges to a unique Nash equilibrium point. In order to do that, we will use the contraction mapping methodology that was invoked in [10] to study the convergence of standard IWFA. However, a fundamental difference between the jammer-free case in [10] and the case considered in the current work is that using a certain transformation, it has been possible in [10] to cast the optimization problem solved in each IWFA iteration as a linear variational inequality (VI). However, when a jammer is present, which is the case considered in this paper, such a transformation is not available, and as we will show below, this will result in a non-linear VI that will require significant manipulation in order to be amenable to applying contraction mapping. In order to proceed with convergence analysis, let

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha_{12} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{1K} & -\alpha_{2K} & \cdots & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \alpha_{21} & \alpha_{31} & \cdots & \alpha_{K1} \\ 0 & 0 & \alpha_{32} & \cdots & \alpha_{K2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{K,K-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \text{and} \quad \beta = \begin{bmatrix} \alpha_{01} \\ \vdots \\ \alpha_{0K} \end{bmatrix}, \tag{14}$$

where we define $\alpha_{jk} \triangleq \left\|[\alpha_{jk}^1, \cdots, \alpha_{jk}^N]\right\|_2$ for all $j \in \{0\} \cup \mathcal{K}$, $k \in \mathcal{K}$, $j \neq k$. Furthermore, for every $k \in \mathcal{K}$, let F_k be a $N \times NK$ block-diagonal matrix whose n-th $1 \times K$ diagonal block is f_k^n. That is,

$$F_k \triangleq \begin{bmatrix} f_k^1 & 0 & \cdots & 0 \\ 0 & f_k^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_k^N \end{bmatrix}, \tag{15}$$

where the i-th entry of f_k^n, $[f_k^n]_i$, $i = 1, \ldots, K$, be defined as follows.

$$[f_k^n]_k = \frac{(S_{\max,0}^n)^2}{(d_{\min,k}^n)^2(c_{\min,k}^n + S_{\max,0}^n)^2} + \frac{\sum_{j=1,\, j\neq k}^K \alpha_{jk}^n S_{\max,j}^n}{(\sum_{j=1,\, j\neq k}^K \alpha_{jk}^n S_{\max,j}^n + \eta_k^n) c_{\min,k}^n d_{\min,k}^n}$$
$$+ \frac{S_{\max,0}^n}{d_{\min,k}^n(c_{\min,k}^n + S_{\max,0}^n)}\left(\frac{1}{d_{\min,k}^n} + \frac{1}{c_{\min,k}^n} + \frac{S_{\max,k}^n}{c_{\min,k}^n(\alpha_{0k}^n c_{\min,k}^n + S_{\max,k}^n)}\right), \tag{16}$$

$$[f_k^n]_i = \frac{(S_{\max,k}^n)^2 d_{\min,k}^n + 2S_{\max,k}^n c_{\min,k}^n(\alpha_{0k}^n c_{\min,k}^n + S_{\max,k}^n)}{d_{\min,k}^n(c_{\min,k}^n)^2(\alpha_{0k}^n c_{\min,k}^n + S_{\max,k}^n)^2}\alpha_{ik}^n$$
$$+ \frac{2S_{\max,0}^n S_{\max,k}^n}{c_{\min,k}^n(\alpha_{0k}^n c_{\min,k}^n + S_{\max,k}^n) d_{\min,k}^n(c_{\min,k}^n + S_{\max,0}^n)}\alpha_{ik}^n, \qquad i \neq k,\ i \in \mathcal{K}, \tag{17}$$

where

$$c_{\min,k}^n = \frac{1}{\alpha_{0k}^n}\left(\sum_{j=1,\, j\neq k}^K \alpha_{jk}^n \eta_j^n + \sigma_k^n\right), \tag{18}$$

$$d_{\min,k}^n = c_{\min,k}^n + \frac{1}{\alpha_{0k}^n}\eta_k^n, \tag{19}$$

with η_k^n being a lower bound on $s_k^{n,\nu}$. That is, for every iteration ν, $\eta_k^n \leq s_k^{n,\nu}$, $\forall k \in \mathcal{K}, n \in \mathcal{N}$. In Appendix B we show that η_k^n is given by

$$\eta_k^n = \left[\frac{1}{N}\big(P_k + \sum_{i=1}^{m_k} \sigma_k^{\pi_k(i)}\big) + \big(\frac{1}{N} - 1\big) \sum_{j=0,\, j\neq k}^K \alpha_{jk}^n S_{\max,j}^n - \sigma_k^{\pi_k(n)}\right]^+, \tag{20}$$

where m_k is the largest integer for which

$$(m_k - 1)(\sigma_k^{\pi_k(j)} + \sum_{i=0,\, i\neq k}^K \alpha_{ik}^{\pi_k(j)} S_{\max,i}^{\pi_k(j)}) \leq P_k + \sum_{i=1}^{m_k-1} \sigma_k^{(i)},$$

is satisfied for all $j \leq m_k$. For each User $k \in \mathcal{K}$ we use $\sigma_k^{(i)}$ to denote the noise variance that satisfies $\sigma_k^{(i)} \leq \sigma_k^{(i+1)}$, for all $i = 1, \ldots, N-1$. We also use $\pi_k(\cdot)$ to denote the tone permutation that satisfy

$$\sigma_k^{\pi_k(1)} + \sum_{\substack{j=0 \\ j\neq k}}^K \alpha_{jk}^{\pi_k(1)} S_{\max,j}^{\pi_k(1)} \leq \cdots \leq \sigma_k^{\pi_k(N)} + \sum_{j=0,\, j\neq k}^K \alpha_{jk}^{\pi_k(N)} S_{\max,j}^{\pi_k(N)}.$$

Theorem 1 (Convergence of GIWFA): Suppose there exists a scalar $\tau \in (0,1)$ such that the following conditions are satisfied

$$\left(1 + \frac{\left\|\sum_{k=1}^K F_k\right\|_2^2}{(1-\tau)^2}\right)(\|A^{-1}B\|_2^2 + \|A^{-1}\beta\|_2^2) < 1, \tag{21}$$

$$\max_n \sum_{k=1}^K \frac{S_{\max,k}^n(2c_{\min,k}^n + \frac{S_{\max,k}^n}{\alpha_{0k}^n})}{(c_{\min,k}^n)^2(c_{\min,k}^n + \frac{S_{\max,k}^n}{\alpha_{0k}^n})^2} \leq \tau + 1, \tag{22}$$

$$\min_n \sum_{k=1}^{K} \Bigg(\frac{(\alpha_{0k}^n)^3 \eta_k^n}{\left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)^2 \left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \eta_k^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)}$$
$$+ \frac{(\alpha_{0k}^n)^3 \eta_k^n}{\left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right) \left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \eta_k^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)^2} \Bigg) \geq 1 - \tau. \tag{23}$$

Then the noncooperative game (7) has a unique Nash equilibrium, and the iterates generated by the GIWFA algorithm converges to this unique equilibrium linearly.

Proof: Fix any equilibrium solution and any starting power allocation. We define the error vector at each iteration to be the difference between the current power allocation and the power allocation at equilibrium. In Appendix A, we show that the conditions (21)–(23) imply the error vectors converge to zero at a geometric rate. Since the choice of equilibrium solution is arbitrary, it follows that the noncooperative game (7) has a unique Nash equilibrium. ■

Notice that the conditions (21)–(23) only depend on the power budget of each user, its maximum allowable power on each tone and the cross-talk coefficients. Hence, these conditions can be used to draw insights into the impact that each of these parameters can have on the system and the users' utilities. In Section IV we will present numerical results that show that for scenarios in which the conditions of Theorem 1 are met, both the users and the jammer converge. We also provide instances showing that the violation of these conditions may cause the algorithm to oscillate. Before we do that, we provide some engineering insights into the convergence conditions in (21)–(23). For instance, let us compare the condition in (21) with the convergence condition of standard IWFA. In order to do that, recall that for IWFA to converge, it is sufficient for the matrices A and B in (14) to satisfy [10]

$$\|A^{-1}B\|_2 < 1. \tag{24}$$

Now, let us assume that (21) is satisfied for some $\tau \in (0,1)$, then this condition can be expressed as

$$\|A^{-1}B\|_2^2 < \frac{1}{1+a} - c \leq 1, \tag{25}$$

where $a = \frac{\|\sum_{k=1}^{K} F_k\|_2^2}{(1-\tau)^2}$ and $c = \|A^{-1}\beta\|_2^2$. It can be seen that (25) implies (24), which indicates that the convergence conditions of GIWFA are, in fact, more stringent than those of standard IWFA.

In order to provide an engineering interpretation of the condition in (22), we observe that each term in the summand on the left hand side of this condition is a monotonically increasing function of $S_{\max,k}^n$. This implies that for (22) to be satisfied, $\{S_{\max,k}^n\}$ have to be relatively small. Now, for given power budgets, $\{P_k\}$, this condition implies that for GIWFA to be guaranteed to converge, each user must not concentrate its power in a small subset of tones, but rather, to distribute its power across many tones. As we will now see, a similar insight can be drawn from the condition in (23).

Condition (23) implies that

$$\min_n \sum_{k=1}^{N} s_k^{n,\nu} \geq \min_n \sum_{k=1}^{N} \eta_k^{n,\nu} > 0. \tag{26}$$

Thus if $s_k^{n,*} \equiv \lim_{\nu\to\infty} s_k^{n,\nu}$, then

$$\min_n \sum_{k=1}^{N} s_k^{n,*} > 0. \tag{27}$$

In words, this says that for guaranteed convergence, every tone n is used by at least one user k. Another insight offered by Theorem 1 is that if the jammer's maximum signal power $S^n_{\max,0}$ on tone n is sufficiently large so that $\eta_k^n = 0$ for all k, then (23) cannot be satisfied and the convergence of the GIWFA is in jeopardy. In order to gain some intuition into this condition, let us consider the scenario in which there is one user, one jammer and two tones, and for ease of exposition, let us ignore the spectral mask. Now, assume that the user allocates all its power on the first tone. In order to minimize the system utility, the jammer updates its power so that it allocates all its power to the tone occupied by the user. Now, if the jammer's power is sufficiently high, the presence of the jammer will force the user to abandon the first tone and to allocate all its power to the second tone (that was previously abandoned). The jammer again updates its power in order to jam the user's signal on the second tone. The user reverts to its initial power allocation, and so on. Hence, one can see that if the user does not occupy all the tones at any iteration, GIWFA may oscillate.

It is worth noting that, based on this insight (which agrees with our numerical experiments), the oscillation mechanism arises because of the jammer's tendency to track the tones on which the users allocate their transmission powers. Now, in standard IWFA, the users compete to increase their individual utilities, but they are not particularly interested in minimizing the overall system utility. That is, while the tracking mechanism is an inherent feature of GIWFA, it is not an inherent feature of standard IWFA. This observation is reflected in the fact that the condition that each tone be occupied by at least one user arises naturally in the case in which a jammer exists, whereas such a condition does not arise in studying the convergence of standard IWFA for scenarios in which no jammer exists; cf. (24).

C. Extension to Asynchronous GIWFA

In Sections III-A and III-B we considered the case in which the users and the jammer update their power allocations sequentially in a predetermined order according to a common clock. However, in many practical scenarios a common clock may not be available for the users and the jammer to operate in such a synchronous fashion. Moreover, even if such a clock is available, due to practical implementation issues, either the users or the jammer may not have access to the most recent multi-user interference. In this case an asynchronous version of the GIWFA algorithm may be more desirable and more robust to implement than a synchronous one.

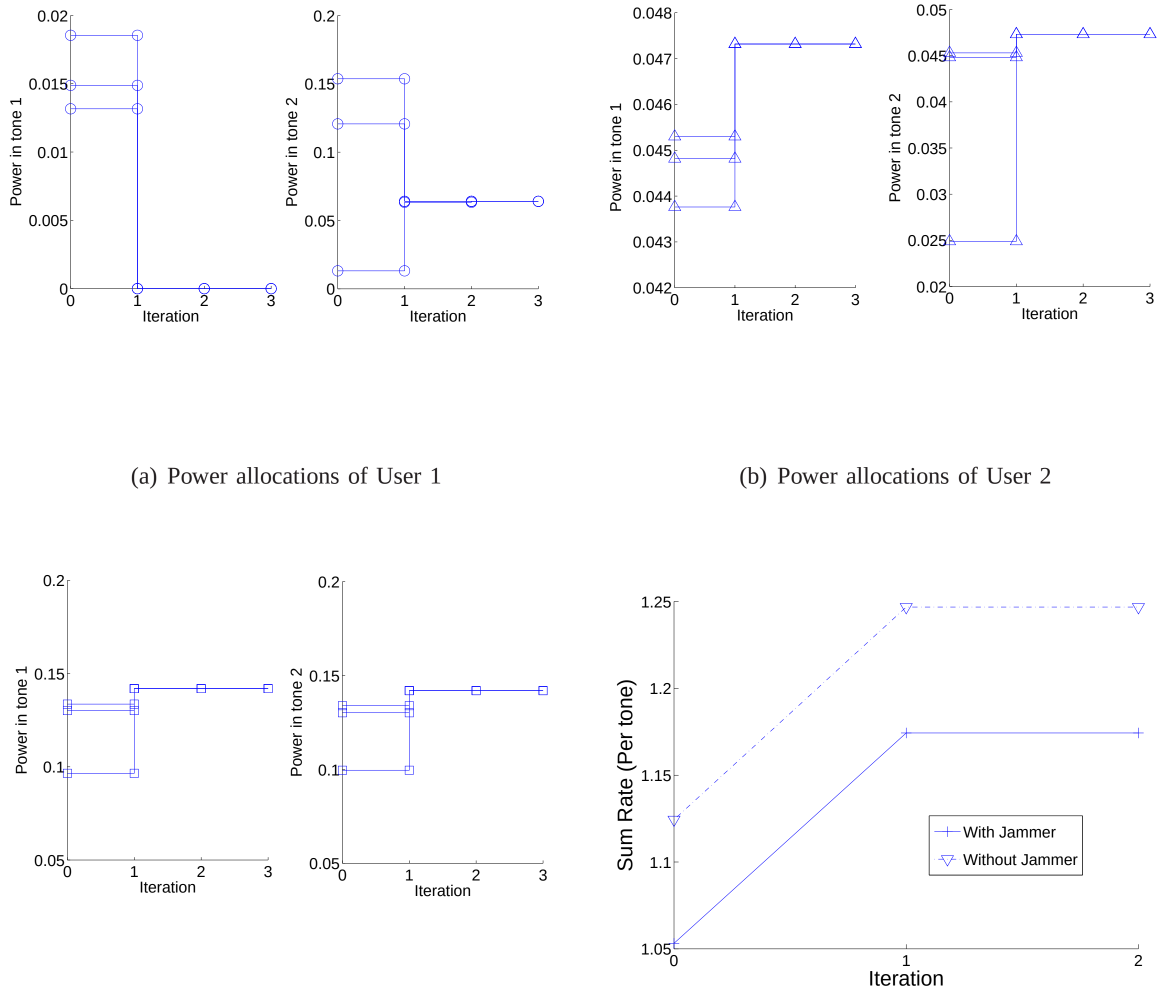
In a totally asynchronous scheme, the users and the jammer update their power allocations at arbitrary time instants using possibly outdated multi-user interference [24]. Under certain mild conditions a fundamental result in [24, Proposition 2.1, Chapter 6] ensures that the asynchronous scheme converges to a unique Nash equilibrium of the game (7) if: 1) each user and the jammer update their power allocations at least once within any sufficiently large, but finite, time interval, and; 2) the iterates contract with respect to some norm. This contraction condition is precisely the same as the set of conditions given in Theorem 1; see also Appendix A.[2] In other words, the conditions given in Theorem 1 ensure convergence of both the synchronous and the asynchronous versions of the GIWFA algorithm.

IV. Numerical Results

In this section we provide a numerical example that illustrates the sufficiency of the conditions given in Theorem 1 for the convergence of the decentralized GIWFA algorithm. We also provide an example that shows that when the conditions in Theorem 1 are violated the users and the jammer may fail to converge and the behaviour of the GIWFA becomes dependent on the initial point. For the numerical examples in this section, the number of users, $K = 4$, and the number of tones $N = 10$, and the maximum allowable power per tone is set to be constant across tones for each user as well as for the jammer; i.e., we set $S^n_{\max,k} = S_{\max,k},\ n = 1, \ldots, 10$ for $k = 0, \ldots, 4$.

Example 1: In this example, the system parameters (i.e, $\alpha^n_{jk}, \sigma^n_k, P_k, S_{\max}, k, \forall\, j \neq k,\ k = 0, \ldots, 4$) are selected at random so as to satisfy the conditions in Theorem 1. The users and the jammer update their power allocations using the GIWFA algorithm described in Section III-A. For this scenario, in Figures 1(a) and 1(b) we plot the power allocations of Users 1 and 2 versus the iteration number for all the tones. For the same scenario, in Figure 1(c) we plot the power allocations of the jammer versus the iteration number. In each of the plots, three randomly chosen allocations were used to initialize the fixed-point algorithm. Since the system parameters were chosen to meet the conditions of Theorem 1,

[2] For the asynchronous scheme the iteration indices ν and $\nu + 1$ in Appendix A ought to be interpreted as the time instants within which each user and the jammer will have updated their power allocations at least once.



(a) Power allocations of User 1

(b) Power allocations of User 2

(c) Power allocations of the jammer

(d) Sum rate with and without a jammer

Fig. 1. The power allocations of Users 1 and 2 are marked by ‘o’ and ‘△’, respectively, whereas the power allocations of the jammer is marked by ‘□’. The GIWFA iterates converge to a unique Nash equilibrium irrespective of the initial power allocation.

the algorithm converges to a unique Nash equilibrium, irrespective of the initial power allocations. In order to quantify the jammer’s impact on the overall system performance, the sum rate of all the users over the ten tones is plotted versus the iteration number in Figure 1(d).

Example 2: In this example, we retain the channel gains of the users as per Example 1. (Since, in Example 1 the gains were selected to meet the conditions in Theorem 1, these gains also meet the IWFA convergence condition (24).) However, the channel gains of the jammer are chosen such that the conditions in Theorem 1 are violated. In this example, we consider two random instances of this scenario. For the

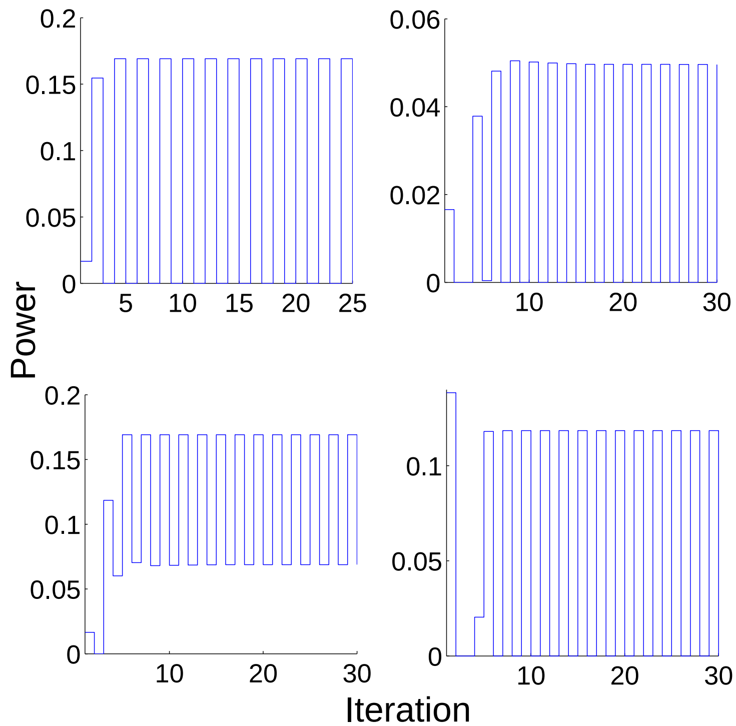


Fig. 2. The power allocations of User 1 do not converge on four different tones.

first instance, we show the power allocations of one of the users on some of the tones. As can be seen from Figure 2, on these tones the user's allocations do not converge, and, in fact, they keep fluctuating. In the second instance of this example we initialize the GIWFA algorithm using three different randomly chosen power allocations. In Figure 3 we plot the sum rate versus the iteration number in this case. It can be seen from that the sum rate fluctuates and no equilibrium is reached.

V. Conclusion

In this paper we considered a communication scenario in which K users and a jammer share N orthogonal tones. We modelled this scenario as a non-cooperative game, and considered an extension of the IWFA algorithm to this problem. We derived sufficient conditions under which the iterates of both synchronous and totally asynchronous decentralized GIWFA algorithms converge to a unique Nash equilibrium. Our theoretical analysis and numerical simulations show that the presence of a strong jammer

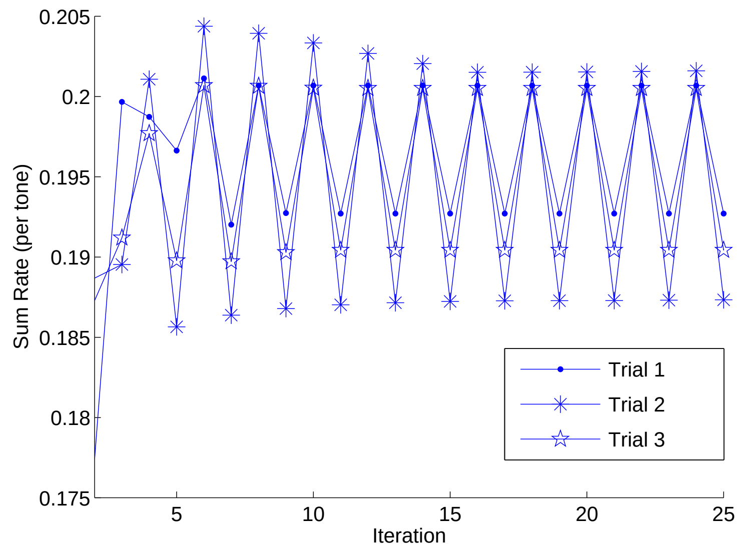


Fig. 3. By changing the initial power allocation, the iterates may oscillate.

can not only reduce the total network throughout, but also cause an otherwise convergent IWFA to oscillate.

Appendix A
Proof of Theorem 1

Recall that we use $s_k^{n,\nu}$ and $s_k^{n,*}$ to denote the power allocated by User $k \in \{0\} \cup \mathcal{K}$ to the n-th tone at the ν-th iteration and at equilibrium, respectively. For the updates of User $k \in \mathcal{K}$, it was shown in [10] that each iteration of the IWFA algorithm in (8) is equivalent to solving the following fixed-point equation.

$$\begin{bmatrix} s_k^{1,\nu} \\ \vdots \\ s_k^{N,\nu} \end{bmatrix} = \begin{bmatrix} s_k^{1,\nu} - \sigma_k^1 - \sum_{j=1}^{k} \alpha_{jk}^1 s_j^{1,\nu} - \sum_{j=k+1}^{K} \alpha_{jk}^1 s_j^{1,\nu-1} - \alpha_{0k}^1 s_0^{1,\nu-1} \\ \vdots \\ s_k^{N,\nu} - \sigma_k^N - \sum_{j=1}^{k} \alpha_{jk}^N s_j^{N,\nu} - \sum_{j=k+1}^{K} \alpha_{jk}^N s_j^{N,\nu-1} - \alpha_{0k}^N s_0^{N,\nu-1} \end{bmatrix}_{\hat{\mathcal{P}}_k}$$

$$= \begin{bmatrix} -\sigma_k^1 - \sum_{j=1}^{k-1} \alpha_{jk}^1 s_j^{1,\nu} - \sum_{j=k+1}^{K} \alpha_{jk}^1 s_j^{1,\nu-1} - \alpha_{0k}^1 s_0^{1,\nu-1} \\ \vdots \\ -\sigma_k^N - \sum_{j=1}^{k-1} \alpha_{jk}^N s_j^{N,\nu} - \sum_{j=k+1}^{K} \alpha_{jk}^N s_j^{N,\nu-1} - \alpha_{0k}^N s_0^{N,\nu-1} \end{bmatrix}_{\hat{\mathcal{P}}_k}, \tag{28}$$

where in (28) we have used that $\alpha_{kk}^n = 1$ for all $n \in \mathcal{N}$, and $[\cdot]_{\hat{\mathcal{P}}_k}$ to denote the projection onto the polyhedron

$$\hat{\mathcal{P}}_k = \left\{(s_k^1, \cdots, s_k^N) | 0 \le s_k^n \le S_{\max,k}^n, n = 1, \ldots, N, \sum_{n=1}^N s_k^n = P_k\right\}. \tag{29}$$

Note that, in contrast with the polyhedron in (6), in the polyhedron in (29), the power constraint is satisfied with equality.

Now, in a similar fashion, the jammer's can update its power in order to solve

$$\begin{bmatrix} s_0^{1,\nu} \\ \vdots \\ s_0^{N,\nu} \end{bmatrix} = \begin{bmatrix} s_0^{1,\nu} + \sum_{k=1}^K \frac{\alpha_{0k}^1 s_k^{1,\nu}}{\left(\sum_{j=1,\, j\neq k}^K \alpha_{jk}^1 s_j^{1,\nu} + \sigma_k^1\right)\left(\sum_{j=1}^K \alpha_{jk}^1 s_j^{1,\nu} + \alpha_{0k}^1 s_0^{1,\nu} + \sigma_k^1\right)} \\ \vdots \\ s_0^{N,\nu} + \sum_{k=1}^K \frac{\alpha_{0k}^N s_k^{N,\nu}}{\left(\sum_{j=1,\, j\neq k}^K \alpha_{jk}^N s_j^{N,\nu} + \sigma_k^N\right)\left(\sum_{j=1}^K \alpha_{jk}^N s_j^{N,\nu} + \alpha_{0k}^N s_0^{N,\nu} + \sigma_k^N\right)} \end{bmatrix}_{\hat{\mathcal{P}}_0}, \tag{30}$$

where the set $\hat{\mathcal{P}}_0$ is defined in a fashion similar to (29).

Let $s_k^{n,*}$ be the power allocation at equilibrium of User $k \in \mathcal{K}$, at tone $n \in \mathcal{N}$. Furthermore, let

$$t_k^{n,\nu} = s_k^{n,\nu} - s_k^{n,*}, \quad \forall k \in \mathcal{K}, \qquad \text{and} \qquad r^{n,\nu} = s_0^{n,\nu} - s_0^{n,*}. \tag{31}$$

At equilibrium we have

$$\begin{bmatrix} s_k^{1,*} \\ \vdots \\ s_k^{N,*} \end{bmatrix} = \begin{bmatrix} -\sigma_k^1 - \sum_{j=1}^{k-1} \alpha_{jk}^1 s_j^{1,*} - \sum_{j=k+1}^K \alpha_{jk}^1 s_j^{1,*} - \alpha_{0k}^1 s_0^{1,*} \\ \vdots \\ -\sigma_k^N - \sum_{j=1}^{k-1} \alpha_{jk}^N s_j^{N,*} - \sum_{j=k+1}^K \alpha_{jk}^N s_j^{N,*} - \alpha_{0k}^N s_0^{N,*} \end{bmatrix}_{\hat{\mathcal{P}}_k}, \tag{32}$$

and

$$\begin{bmatrix} s_0^{1,*} \\ \vdots \\ s_0^{N,*} \end{bmatrix} = \begin{bmatrix} s_0^{1,*} + \sum_{k=1}^K \frac{\alpha_{0k}^1 s_k^{1,*}}{\left(\sum_{j=1,\, j\neq k}^K \alpha_{jk}^1 s_j^{1,*} + \sigma_k^1\right)\left(\sum_{j=1}^K \alpha_{jk}^1 s_j^{1,*} + \alpha_{0k}^1 s_0^{1,*} + \sigma_k^1\right)} \\ \vdots \\ s_0^{N,*} + \sum_{k=1}^K \frac{\alpha_{0k}^N s_k^{N,*}}{\left(\sum_{j=1,\, j\neq k}^K \alpha_{jk}^N s_j^{N,*} + \sigma_k^N\right)\left(\sum_{j=1}^K \alpha_{jk}^N s_j^{N,*} + \alpha_{0k}^N s_0^{N,*} + \sigma_k^N\right)} \end{bmatrix}_{\hat{\mathcal{P}}_0}. \tag{33}$$

We now subtract (32) from (28), and (33) from (30). Using the non-expansiveness property of the projection operator [10], one can write

$$\begin{aligned} \left\| \begin{bmatrix} t_k^{1,\nu} \\ \vdots \\ t_k^{N,\nu} \end{bmatrix} \right\| &\le \left\| \begin{bmatrix} -\alpha_{0k}^1 r^{1,\nu-1} - \sum_{j=1}^{k-1} \alpha_{jk}^1 t_j^{1,\nu} - \sum_{j=k+1}^K \alpha_{jk}^1 t_j^{1,\nu-1} \\ \vdots \\ -\alpha_{0k}^N r^{N,\nu-1} - \sum_{j=1}^{k-1} \alpha_{jk}^N t_j^{N,\nu} - \sum_{j=k+1}^K \alpha_{jk}^N t_j^{N,\nu-1} \end{bmatrix} \right\| \\ &\le \left\| \begin{bmatrix} \alpha_{0k}^1 r^{1,\nu-1} \\ \vdots \\ \alpha_{0k}^N r^{N,\nu-1} \end{bmatrix} \right\| + \left\| \sum_{j=1}^{k-1} \begin{bmatrix} \alpha_{jk}^1 t_j^{1,\nu} \\ \vdots \\ \alpha_{jk}^N t_j^{N,\nu} \end{bmatrix} \right\| + \left\| \sum_{j=k+1}^{K} \begin{bmatrix} \alpha_{jk}^1 t_j^{1,\nu-1} \\ \vdots \\ \alpha_{jk}^N t_j^{N,\nu-1} \end{bmatrix} \right\| \end{aligned}$$

$$\leq \alpha_{0,k}\|r^{\nu-1}\| + \sum_{j=1}^{k-1}\alpha_{jk}\|t_j^{\nu}\| + \sum_{j=k+1}^{K}\alpha_{jk}\|t_j^{\nu-1}\|, \tag{34}$$

where in (34) we have used t_k^{ν} and r^{ν} to denote the vectors $[t_k^{1,\nu}, \cdots, t_k^{N,\nu}]^T$ and $[r^{1,\nu}, \cdots, r^{N,\nu}]^T$, respectively, and α_{jk} to denote $\big\|[\alpha_{jk}^1, \cdots, \alpha_{jk}^N]\big\|_2$.

Using a technique similar to the one in [10] we can express the inequalities in (34) for all users simultaneously in the following matrix form.

$$A\begin{bmatrix}\|t_1^{\nu}\|\\ \vdots\\ \|t_K^{\nu}\|\end{bmatrix} \leq \begin{bmatrix}B & \beta\end{bmatrix}\begin{bmatrix}\|t_1^{\nu-1}\|\\ \vdots\\ \|t_K^{\nu-1}\|\\ \|r^{\nu-1}\|\end{bmatrix}, \tag{35}$$

where

$$A = \begin{bmatrix}1 & 0 & \cdots & 0\\ -\alpha_{12} & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\alpha_{1K} & -\alpha_{2K} & \cdots & 1\end{bmatrix}, \quad B = \begin{bmatrix}0 & \alpha_{21} & \alpha_{31} & \cdots & \alpha_{K1}\\ 0 & 0 & \alpha_{32} & \cdots & \alpha_{K2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \alpha_{K,K-1}\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}, \quad \text{and} \quad \beta = \begin{bmatrix}\alpha_{01}\\ \vdots\\ \alpha_{0K}\end{bmatrix}, \tag{36}$$

and the inequality in (35) is to be interpreted element-wise. Notice that A is a non-singular Z matrix with (entry-wise) non-negative inverse. The matrix B and the vector β are also non-negative. Hence using [25, Property 2.5.3.18], we have that (35) imply that

$$\begin{bmatrix}\|t_1^{\nu}\|\\ \vdots\\ \|t_K^{\nu}\|\end{bmatrix} \leq \begin{bmatrix}A^{-1}B & A^{-1}\beta\end{bmatrix}\begin{bmatrix}\|t_1^{\nu-1}\|\\ \vdots\\ \|t_K^{\nu-1}\|\\ \|r^{\nu-1}\|\end{bmatrix}. \tag{37}$$

If we use t^{ν} to denote the vector $\begin{bmatrix}\|t_1^{\nu}\| & \cdots & \|t_K^{\nu}\|\end{bmatrix}^T$, then (37) implies that

$$\|t^{\nu}\| \leq \begin{bmatrix}\|A^{-1}B\|_2 & \|A^{-1}\beta\|\end{bmatrix}\begin{bmatrix}\|t^{\nu-1}\|\\ \|r^{\nu-1}\|\end{bmatrix}. \tag{38}$$

We now turn our attention to the jammer's updates; cf. (30). In order to simplify our exposition, we will use the following notation.

$$c_k^{n,*} = \tfrac{1}{\alpha_{0k}^n}\sum_{j=1,\, j\neq k}^{K}\alpha_{jk}^n s_j^{n,*} + \tfrac{\sigma_k^n}{\alpha_{0k}^n},$$

$$c_k^{n,\nu} = \frac{1}{\alpha_{0k}^n} \sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n s_j^{n,\nu} + \frac{\sigma_k^n}{\alpha_{0k}^n}, \tag{39}$$

$$d_k^{n,*} = c_k^{n,*} + \frac{s_k^{n,*}}{\alpha_{0k}^n},$$

$$d_k^{n,\nu} = c_k^{n,\nu} + \frac{s_k^{n,\nu}}{\alpha_{0k}^n}. \tag{40}$$

Using a technique similar to the one used for the users' updates and employing the non-expansiveness property of the projection operator, we use (30) to write

$$\left\| \begin{bmatrix} r^{1,\nu} \\ \vdots \\ r^{N,\nu} \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} r^{1,\nu} + \sum_{k=1}^{K} \frac{s_k^{1,\nu}}{(c_k^{1,\nu}+s_0^{1,\nu})(d_k^{1,\nu}+s_0^{1,\nu})} - \frac{s_k^{1,*}}{(c_k^{1,*}+s_0^{1,*})(d_k^{1,*}+s_0^{1,*})} \\ \vdots \\ r^{N,\nu} + \sum_{k=1}^{K} \frac{s_k^{N,\nu}}{(c_k^{N,\nu}+s_0^{N,\nu})(d_k^{N,\nu}+s_0^{N,\nu})} - \frac{s_k^{N,*}}{(c_k^{N,*}+s_0^{N,*})(d_k^{N,*}+s_0^{N,*})} \end{bmatrix} \right\|. \tag{41}$$

Using partial fraction expansion, the n-th entry of the vector on the right hand side of (41) can be written as

$$r^{n,\nu}\left(1 - \sum_{k=1}^{K}\left(\frac{\alpha_{0k}^n}{(c_k^{n,\nu}+s_0^{n,*})(c_k^{n,\nu}+s_0^{n,*}+r^{n,\nu})} - \frac{\alpha_{0k}^n}{(d_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*}+r^{n,\nu})}\right)\right)$$
$$- \sum_{k=1}^{K}\left(\frac{s_k^{n,*}}{(c_k^{n,*}+s_0^{n,*})(d_k^{n,*}+s_0^{n,*})} - \frac{s_k^{n,\nu}}{(c_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*})}\right). \tag{42}$$

Let Υ^ν be an $N \times N$ diagonal matrix with the n-th diagonal entry given by

$$\left(1 - \sum_{k=1}^{K}\left(\frac{\alpha_{0k}^n}{(c_k^{n,\nu}+s_0^{n,*})(c_k^{n,\nu}+s_0^{n,*}+r^{n,\nu})} - \frac{\alpha_{0k}^n}{(d_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*}+r^{n,\nu})}\right)\right). \tag{43}$$

Furthermore, let γ_k^ν be an N-dimensional vector whose n-th entry is given by

$$\gamma_k^{n,\nu} = \left|\frac{s_k^{n,*}}{(c_k^{n,*}+s_0^{n,*})(d_k^{n,*}+s_0^{n,*})} - \frac{s_k^{n,\nu}}{(c_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*})}\right|. \tag{44}$$

Now, (41) can be bounded as follows

$$\|r^\nu\| \leq \|\Upsilon^\nu r^\nu\| + \left\|\sum_{k=1}^{K}\gamma_k^\nu\right\|$$
$$\leq \|\Upsilon^\nu\|_2 \|r^\nu\| + \left\|\sum_{k=1}^{K}\gamma_k^\nu\right\|.$$

Assuming that $\|\Upsilon^\nu\|_2 < 1$, then we have

$$\|r^\nu\| \leq \left(1 - \|\Upsilon^\nu\|_2\right)^{-1}\left\|\sum_{k=1}^{K}\gamma_k^\nu\right\|. \tag{45}$$

In order to analyze the matrix Υ^ν and the vectors $\{\gamma_k^\nu\}$, we will need a lower bound on $s_k^{n,\nu}$. In Appendix B we provide a lower bound η_k^n such that $0 \leq \eta_k^n \leq s_k^{n,\nu}$, for all iterations ν, $k \in \mathcal{K}$, $n \in \mathcal{N}$.

Using this value of η_k^n, we can readily derive a lower bound on $c_k^{n,\nu}$. In particular, if we let $c_{\min,k}^n$ denote this bound, then it follows from (39) that

$$c_k^{n,\nu} = \tfrac{1}{\alpha_{0k}^n}\sum_{j=1,\,j\neq k}^{K}\alpha_{jk}^n s_j^{n,\nu} + \tfrac{\sigma_k^n}{\alpha_{0k}^n} \geq \frac{1}{\alpha_{0k}^n}\left(\sum_{j=1,\,j\neq k}^{K}\alpha_{jk}^n\eta_j^n + \sigma_k^n\right) \triangleq c_{\min,k}^n. \tag{46}$$

Similarly, a lower bound on $d_k^{n,\nu}$ can be derived from (40)

$$d_{\min,k}^n \triangleq c_{\min,k}^n + \frac{1}{\alpha_{0k}^n}\eta_k^n. \tag{47}$$

Now that we have a lower bound on $s_k^{n,\nu}$, $c_k^{n,\nu}$ and $d_k^{n,\nu}$, we can proceed to analyze $\gamma_k^{n,\nu}$ in (44). From (44), we have Our goal is to bound $\{\gamma_k^{n,\nu}\}$ as a linear combination of $|t_j^{n,\nu}|_{j=1}^K$. This requires some detailed computation which we present below. By definition, we have

$$\begin{aligned}
\gamma_k^{n,\nu} &= \frac{|s_k^{n,*}(c_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*}) - s_k^{n,\nu}(c_k^{n,*}+s_0^{n,*})(d_k^{n,*}+s_0^{n,*})|}{(c_k^{n,*}+s_0^{n,*})(d_k^{n,*}+s_0^{n,*})(c_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*})}\\
&\leq \frac{(s_0^{n,*})^2|s_k^{n,*}-s_k^{n,\nu}| + s_0^{n,*}\left|s_k^{n,*}(c_k^{n,\nu}+d_k^{n,\nu}) - s_k^{n,\nu}(c_k^{n,*}+d_k^{n,*})\right|}{(c_k^{n,*}+s_0^{n,*})(d_k^{n,*}+s_0^{n,*})(c_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*})}\\
&\qquad + \frac{\left|s_k^{n,*}c_k^{n,\nu}d_k^{n,\nu} - s_k^{n,\nu}c_k^{n,*}d_k^{n,*}\right|}{(c_k^{n,*}+s_0^{n,*})(d_k^{n,*}+s_0^{n,*})(c_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*})}\\
&\leq \frac{(s_0^{n,*})^2|s_k^{n,*}-s_k^{n,\nu}|}{d_k^{n,\nu}d_k^{n,*}(c_k^{n,*}+s_0^{n,*})(c_k^{n,\nu}+s_0^{n,*})} + \frac{s_0^{n,*}\left|s_k^{n,*}(c_k^{n,\nu}+d_k^{n,\nu}) - s_k^{n,\nu}(c_k^{n,*}+d_k^{n,*})\right|}{c_k^{n,*}d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+s_0^{n,*})}\\
&\qquad + \frac{\left|s_k^{n,*}c_k^{n,\nu}d_k^{n,\nu} - s_k^{n,\nu}c_k^{n,*}d_k^{n,*}\right|}{(c_k^{n,*}+s_0^{n,*})(d_k^{n,*}+s_0^{n,*})(c_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*})} &(48)\\
&\leq \frac{(S_{\max,0}^n)^2|s_k^{n,*}-s_k^{n,\nu}|}{d_k^{n,\nu}d_k^{n,*}(c_k^{n,*}+S_{\max,0}^n)(c_k^{n,\nu}+S_{\max,0}^n)} + \frac{S_{\max,0}^n\left|s_k^{n,*}(c_k^{n,\nu}+d_k^{n,\nu}) - s_k^{n,\nu}(c_k^{n,*}+d_k^{n,*})\right|}{c_k^{n,*}d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S_{\max,0}^n)}\\
&\qquad + \frac{\left|s_k^{n,*}c_k^{n,\nu}d_k^{n,\nu} - s_k^{n,\nu}c_k^{n,*}d_k^{n,*}\right|}{(c_k^{n,*}+s_0^{n,*})(d_k^{n,*}+s_0^{n,*})(c_k^{n,\nu}+s_0^{n,*})(d_k^{n,\nu}+s_0^{n,*})} &(49)\\
&\leq \frac{(S_{\max,0}^n)^2|t_k^{n,\nu}|}{(d_{\min,k}^n)^2(c_{\min,k}^n+S_{\max,0}^n)^2} + \frac{S_{\max,0}^n\left|s_k^{n,*}(c_k^{n,\nu}+d_k^{n,\nu}) - s_k^{n,\nu}(c_k^{n,*}+d_k^{n,*})\right|}{c_k^{n,*}d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S_{\max,0}^n)}\\
&\qquad + \frac{\left|s_k^{n,*}c_k^{n,\nu}d_k^{n,\nu} - s_k^{n,\nu}c_k^{n,*}d_k^{n,*}\right|}{c_k^{n,*}d_k^{n,*}c_k^{n,\nu}d_k^{n,\nu}}, &(50)
\end{aligned}$$

where in (49) we have used the fact that both the first and the second term of (48) are monotone increasing in $s_0^{n,*}$.

Next we bound the third and second term in (50) separately. Let $a_k^{n,\nu}$ denote the third term of (50). Then, using the definition of $d_k^{n,\nu}$ in (40), we obtain

$$a_k^{n,\nu} = \frac{\left|s_k^{n,*}c_k^{n,\nu}d_k^{n,\nu} - s_k^{n,\nu}c_k^{n,*}d_k^{n,*}\right|}{c_k^{n,*}d_k^{n,*}c_k^{n,\nu}d_k^{n,\nu}}$$

$$
\begin{aligned}
&= \frac{\left|s_k^{n,*}c_k^{n,\nu}(c_k^{n,\nu}+\frac{s_k^{n,\nu}}{\alpha_{0k}^n}) - s_k^{n,\nu}c_k^{n,*}(c_k^{n,*}+\frac{s_k^{n,*}}{\alpha_{0k}^n})\right|}{c_k^{n,*}d_k^{n,*}c_k^{n,\nu}d_k^{n,\nu}} \\
&= \frac{\left|\frac{s_k^{n,*}s_k^{n,\nu}}{\alpha_{0k}^n}(c_k^{n,\nu}-c_k^{n,*}) + s_k^{n,*}(c_k^{n,\nu})^2 - s_k^{n,\nu}(c_k^{n,*})^2\right|}{c_k^{n,*}d_k^{n,*}c_k^{n,\nu}d_k^{n,\nu}} \\
&\le \frac{\frac{s_k^{n,*}s_k^{n,\nu}}{\alpha_{0k}^n}|c_k^{n,\nu}-c_k^{n,*}| + |s_k^{n,*}(c_k^{n,\nu})^2 - (s_k^{n,*}+t_k^{n,\nu})(c_k^{n,*})^2|}{c_k^{n,*}d_k^{n,*}c_k^{n,\nu}d_k^{n,\nu}} \\
&\le \frac{\frac{s_k^{n,*}s_k^{n,\nu}}{\alpha_{0k}^n}|c_k^{n,\nu}-c_k^{n,*}| + s_k^{n,*}|(c_k^{n,\nu})^2 - (c_k^{n,*})^2| + |t_k^{n,\nu}|(c_k^{n,*})^2}{c_k^{n,*}d_k^{n,*}c_k^{n,\nu}d_k^{n,\nu}} \\
&= \frac{s_k^{n,*}s_k^{n,\nu}|c_k^{n,\nu}-c_k^{n,*}|}{\alpha_{0k}^n c_k^{n,*}c_k^{n,\nu}(c_k^{n,*}+\frac{s_k^{n,*}}{\alpha_{0k}^n})(c_k^{n,\nu}+\frac{s_k^{n,\nu}}{\alpha_{0k}^n})} + \frac{s_k^{n,*}(c_k^{n,\nu}+c_k^{n,*})|c_k^{n,\nu}-c_k^{n,*}|}{c_k^{n,*}c_k^{n,\nu}(c_k^{n,*}+\frac{s_k^{n,*}}{\alpha_{0k}^n})d_k^{n,\nu}} + \frac{|t_k^{n,\nu}|c_k^{n,*}}{d_k^{n,*}c_k^{n,\nu}d_k^{n,\nu}} \qquad (51) \\
&\le \frac{(S_{\max,k}^n)^2|c_k^{n,\nu}-c_k^{n,*}|}{\alpha_{0k}^n(c_{\min,k}^n)^2(c_{\min,k}^n+\frac{S_{\max,k}^n}{\alpha_{0k}^n})^2} + \frac{S_{\max,k}^n(c_k^{n,\nu}+c_k^{n,*})|c_k^{n,\nu}-c_k^{n,*}|}{c_k^{n,*}c_k^{n,\nu}(c_k^{n,*}+\frac{S_{\max,k}^n}{\alpha_{0k}^n})d_k^{n,\nu}} + \frac{|t_k^{n,\nu}|c_k^{n,*}}{d_k^{n,*}c_k^{n,\nu}d_k^{n,\nu}} \qquad (52) \\
&= \frac{(S_{\max,k}^n)^2|c_k^{n,\nu}-c_k^{n,*}|}{\alpha_{0k}^n(c_{\min,k}^n)^2(c_{\min,k}^n+\frac{S_{\max,k}^n}{\alpha_{0k}^n})^2} + \frac{S_{\max,k}^n|c_k^{n,\nu}-c_k^{n,*}|}{c_k^{n,\nu}(c_k^{n,*}+\frac{S_{\max,k}^n}{\alpha_{0k}^n})d_k^{n,\nu}} + \frac{S_{\max,k}^n|c_k^{n,\nu}-c_k^{n,*}|}{c_k^{n,*}(c_k^{n,*}+\frac{S_{\max,k}^n}{\alpha_{0k}^n})d_k^{n,\nu}} \\
&\qquad + \frac{|t_k^{n,\nu}|c_k^{n,*}}{(c_k^{n,*}+\frac{s_k^{n,*}}{\alpha_{0k}^n})c_k^{n,\nu}d_k^{n,\nu}} \\
&\le \frac{(S_{\max,k}^n)^2|c_k^{n,\nu}-c_k^{n,*}|}{\alpha_{0k}^n(c_{\min,k}^n)^2(c_{\min,k}^n+\frac{S_{\max,k}^n}{\alpha_{0k}^n})^2} + \frac{2S_{\max,k}^n|c_k^{n,\nu}-c_k^{n,*}|}{c_{\min,k}^n d_{\min,k}^n(c_{\min,k}^n+\frac{S_{\max,k}^n}{\alpha_{0k}^n})} + \frac{|t_k^{n,\nu}|c_k^{n,*}}{(c_k^{n,*}+\frac{s_k^{n,*}}{\alpha_{0k}^n})c_{\min,k}^n d_{\min,k}^n} \qquad (53) \\
&\le \left(\frac{(S_{\max,k}^n)^2}{(\alpha_{0k}^n)^2(c_{\min,k}^n)^2(c_{\min,k}^n+\frac{S_{\max,k}^n}{\alpha_{0k}^n})^2} + \frac{2S_{\max,k}^n}{\alpha_{0k}^n c_{\min,k}^n d_{\min,k}^n(c_{\min,k}^n+\frac{S_{\max,k}^n}{\alpha_{0k}^n})}\right)\sum_{j=1,\,j\neq k}^K \alpha_{jk}^n|t_j^{n,\nu}| \\
&\qquad + \frac{|t_k^{n,\nu}|\sum_{j=1,\,j\neq k}^K \alpha_{jk}^n S_{\max,j}^n}{(\sum_{j=1,\,j\neq k}^K \alpha_{jk}^n S_{\max,j}^n + s_k^{n,*})c_{\min,k}^n d_{\min,k}^n}, \qquad (54) \\
&\le \frac{(S_{\max,k}^n)^2 d_{\min,k}^n + 2S_{\max,k}^n c_{\min,k}^n(\alpha_{0k}^n c_{\min,k}^n + S_{\max,k}^n)}{d_{\min,k}^n(c_{\min,k}^n)^2(\alpha_{0k}^n c_{\min,k}^n + S_{\max,k}^n)^2}\sum_{j=1,\,j\neq k}^K \alpha_{jk}^n|t_j^{n,\nu}| \\
&\qquad + \frac{\sum_{j=1,\,j\neq k}^K \alpha_{jk}^n S_{\max,j}^n}{(\sum_{j=1,\,j\neq k}^K \alpha_{jk}^n S_{\max,j}^n + \eta_k^n)c_{\min,k}^n d_{\min,k}^n}|t_k^{n,\nu}|, \qquad (55)
\end{aligned}
$$

where in (52), we have used that the first term in (51) is monotonically increasing in both $s_k^{n,*}$ and $s_k^{n,\nu}$, and that the second term in (51) is monotonically increasing in $s_k^{n,*}$. Similarly, in (54) we have used the fact that in (53), the last term is monotonically increasing in $c_k^{n,*}$.

We now consider the second term in (50). Denoting this term by $b_k^{n,\nu}$, we have,

$$\begin{aligned}
b_k^{n,\nu} &= \frac{S^n_{\max,0}\big|s_k^{n,*}(c_k^{n,\nu}+d_k^{n,\nu})-s_k^{n,\nu}(c_k^{n,*}+d_k^{n,*})\big|}{c_k^{n,*}d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}\\
&= \frac{S^n_{\max,0}\big|s_k^{n,*}(c_k^{n,\nu}+d_k^{n,\nu})-(s_k^{n,*}+t_k^{n,\nu})(c_k^{n,*}+d_k^{n,*})\big|}{c_k^{n,*}d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}\\
&= \frac{S^n_{\max,0}\big|s_k^{n,*}(c_k^{n,\nu}+d_k^{n,\nu}-c_k^{n,*}-d_k^{n,*})-t_k^{n,\nu}(c_k^{n,*}+d_k^{n,*})\big|}{c_k^{n,*}d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}\\
&\le \frac{S^n_{\max,0}s_k^{n,*}|c_k^{n,\nu}+d_k^{n,\nu}-c_k^{n,*}-d_k^{n,*}|}{c_k^{n,*}d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}+\frac{S^n_{\max,0}|t_k^{n,\nu}|(c_k^{n,*}+d_k^{n,*})}{c_k^{n,*}d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}\\
&= \frac{S^n_{\max,0}s_k^{n,*}|c_k^{n,\nu}+d_k^{n,\nu}-c_k^{n,*}-d_k^{n,*}|}{c_k^{n,*}(c_k^{n,*}+\frac{s_k^{n,*}}{\alpha^n_{0k}})d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}+\frac{S^n_{\max,0}|t_k^{n,\nu}|}{d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}+\frac{S^n_{\max,0}|t_k^{n,\nu}|}{c_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}\\
&\le \frac{S^n_{\max,0}s_k^{n,*}\big(2\sum_{j=1,\,j\neq k}^K\alpha^n_{jk}|t_j^{n,\nu}|+|t_k^{n,\nu}|\big)}{\alpha^n_{0k}c_k^{n,*}(c_k^{n,*}+\frac{s_k^{n,*}}{\alpha^n_{0k}})d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}+\frac{S^n_{\max,0}|t_k^{n,\nu}|}{d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}+\frac{S^n_{\max,0}|t_k^{n,\nu}|}{c_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}
\end{aligned} \tag{56}$$

$$\le \frac{S^n_{\max,0}S^n_{\max,k}\big(2\sum_{j=1,\,j\neq k}^K\alpha^n_{jk}|t_j^{n,\nu}|+|t_k^{n,\nu}|\big)}{c_k^{n,*}(\alpha^n_{0k}c_k^{n,*}+S^n_{\max,k})d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}+\frac{S^n_{\max,0}|t_k^{n,\nu}|}{d_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})}+\frac{S^n_{\max,0}|t_k^{n,\nu}|}{c_k^{n,*}d_k^{n,\nu}(c_k^{n,\nu}+S^n_{\max,0})} \tag{57}$$

$$\begin{aligned}
&\le \frac{S^n_{\max,0}S^n_{\max,k}\big(2\sum_{j=1,\,j\neq k}^K\alpha^n_{jk}|t_j^{n,\nu}|+|t_k^{n,\nu}|\big)}{c^n_{\min,k}(\alpha^n_{0k}c^n_{\min,k}+S^n_{\max,k})d^n_{\min,k}(c^n_{\min,k}+S^n_{\max,0})}+\frac{S^n_{\max,0}}{d^n_{\min,k}(c^n_{\min,k}+S^n_{\max,0})}\Big(\frac{1}{d^n_{\min,k}}+\frac{1}{c^n_{\min,k}}\Big)|t_k^{n,\nu}|\\
&= \frac{2S^n_{\max,0}S^n_{\max,k}}{c^n_{\min,k}(\alpha^n_{0k}c^n_{\min,k}+S^n_{\max,k})d^n_{\min,k}(c^n_{\min,k}+S^n_{\max,0})}\sum_{j=1,\,j\neq k}^K\alpha^n_{jk}|t_j^{n,\nu}|\\
&\quad+\frac{S^n_{\max,0}}{d^n_{\min,k}(c^n_{\min,k}+S^n_{\max,0})}\left(\frac{1}{d^n_{\min,k}}+\frac{1}{c^n_{\min,k}}+\frac{S^n_{\max,k}}{c^n_{\min,k}(\alpha^n_{0k}c^n_{\min,k}+S^n_{\max,k})}\right)|t_k^{n,\nu}|,
\end{aligned} \tag{58}$$

where in (57), we have used that the first term in (56) is monotonically increasing in $s_k^{n,*}$.

Using the bounds on $a_k^{n,\nu}$ and $b_k^{n,\nu}$ in (55) and (58), respectively, the scalar $\gamma_k^{n,\nu}$ in (50) can be now bounded by a linear combination of $\{|t_j^{n,\nu}|\}_{j=1}^K$. In particular, let f_k^n be a $1\times K$ row vector whose entries are defined as,

$$\begin{aligned}
[f_k^n]_k &= \frac{(S^n_{\max,0})^2}{(d^n_{\min,k})^2(c^n_{\min,k}+S^n_{\max,0})^2}+\frac{\sum_{j=1,\,j\neq k}^K\alpha^n_{jk}S^n_{\max,j}}{(\sum_{j=1,\,j\neq k}^K\alpha^n_{jk}S^n_{\max,j}+\eta^n_k)c^n_{\min,k}d^n_{\min,k}}\\
&\quad+\frac{S^n_{\max,0}}{d^n_{\min,k}(c^n_{\min,k}+S^n_{\max,0})}\left(\frac{1}{d^n_{\min,k}}+\frac{1}{c^n_{\min,k}}+\frac{S^n_{\max,k}}{c^n_{\min,k}(\alpha^n_{0k}c^n_{\min,k}+S^n_{\max,k})}\right),
\end{aligned} \tag{59}$$

$$[f_k^n]_i = \frac{(S^n_{\max,k})^2d^n_{\min,k}+2S^n_{\max,k}c^n_{\min,k}(\alpha^n_{0k}c^n_{\min,k}+S^n_{\max,k})}{d^n_{\min,k}(c^n_{\min,k})^2(\alpha^n_{0k}c^n_{\min,k}+S^n_{\max,k})^2}\alpha^n_{ik}$$

$$+\frac{2S^n_{\max,0}S^n_{\max,k}}{c^n_{\min,k}(\alpha^n_{0k}c^n_{\min,k}+S^n_{\max,k})d^n_{\min,k}(c^n_{\min,k}+S^n_{\max,0})}\alpha^n_{ik}, \qquad i\neq k, \tag{60}$$

and let

$$\underline{t}^{n,\nu}=[|t^{n,\nu}_1|,\cdots,|t^{n,\nu}_K|]^T. \tag{61}$$

Using (59) and (60), $\gamma^{n,\nu}_k$ can be now bounded by

$$\gamma^{n,\nu}_k \leq f^n_k\underline{t}^{n,\nu}. \tag{62}$$

Hence, the vector γ^{ν}_k can be element-wise bounded by the product of an $N\times NK$ block-diagonal matrix, F_k and a $KN\times 1$ vector whose entries are $|t^{n,\nu}_j|, n=1,\ldots,N, k=1,\ldots,K$. In particular, we define

$$F_k \triangleq \begin{bmatrix} f^1_k & 0 & \cdots & 0\\ 0 & f^2_k & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & f^N_k \end{bmatrix}, \tag{63}$$

and write

$$\gamma^{\nu}_k \leq F_k t^{\nu}, \tag{64}$$

where t^{ν} is defined as

$$t^{\nu}\triangleq\begin{bmatrix}\underline{t}^{1,\nu}\\ \vdots\\ \underline{t}^{N,\nu}\end{bmatrix}. \tag{65}$$

Substituting from (64) into (45), we obtain

$$\|r^{\nu}\| \leq (1-\|\Upsilon^{\nu}\|_2)^{-1}\Big\|\sum_{k=1}^{K}F_k\Big\|_2\|t^{\nu}\|. \tag{66}$$

Now using (38), we have

$$\|r^{\nu}\| \leq (1-\|\Upsilon^{\nu}\|_2)^{-1}\Big\|\sum_{k=1}^{K}F_k\Big\|_2\begin{bmatrix}\|A^{-1}B\|_2 & \|A^{-1}\beta\|\end{bmatrix}\begin{bmatrix}\|t^{\nu-1}\|\\ \|r^{\nu-1}\|\end{bmatrix}. \tag{67}$$

Writing (66) along with (38) in a vector form yields

$$\begin{bmatrix}\|t^{\nu}\|\\ \|r^{\nu}\|\end{bmatrix} \leq \begin{bmatrix}1 & 0\\ 0 & (1-\|\Upsilon^{\nu}\|_2)^{-1}\Big\|\sum_{k=1}^{K}F_k\Big\|_2\end{bmatrix}\begin{bmatrix}\|A^{-1}B\|_2 & \|A^{-1}\beta\|\\ \|A^{-1}B\|_2 & \|A^{-1}\beta\|\end{bmatrix}\begin{bmatrix}\|t^{\nu-1}\|\\ \|r^{\nu-1}\|\end{bmatrix}, \tag{68}$$

where the inequality is to be interpreted element-wise. A sufficient condition for convergence is to have

$$\left\|\begin{bmatrix}1 & 0\\ 0 & (1-\|\Upsilon^{\nu}\|_2)^{-1}\Big\|\sum_{k=1}^{K}F_k\Big\|_2\end{bmatrix}\begin{bmatrix}\|A^{-1}B\|_2 & \|A^{-1}\beta\|\\ \|A^{-1}B\|_2 & \|A^{-1}\beta\|\end{bmatrix}\right\|_2 < 1. \tag{69}$$

In Appendix C, we show that the condition in (69) is equivalent to the condition that

$$\Big(1 + (1 - \|\Upsilon^\nu\|_2)^{-2}\Big\|\sum_{k=1}^{K} F_k\Big\|_2^2\Big)\big(\|A^{-1}B\|_2 + \|A^{-1}\beta\|\big) < 1. \tag{70}$$

Now, $\|\Upsilon^\nu\|_2$ is the only iteration-dependent entry in (70). Observe that the left hand side of (70) is a monotone increasing function of $\|\Upsilon^\nu\|_2$. Hence, for (70) to hold, it is sufficient to have

$$\|\Upsilon^\nu\|_2 \leq \tau, \tag{71}$$

where τ is an iteration-independent constant, that satisfies

$$\Big(1 + (1 - \tau)^{-2}\Big\|\sum_{k=1}^{K} F_k\Big\|_2^2\Big)\big(\|A^{-1}B\|_2 + \|A^{-1}\beta\|\big) < 1 \tag{72}$$

We now consider the diagonal matrix Υ^ν; cf. (43). The spectral norm of this matrix is given by the maximum absolute value of its diagonal entries. Hence, in order to satisfy (71), we must have

$$\max_n\left(1 - \sum_{k=1}^{K}\left(\frac{\alpha_{0k}^n}{(c_k^{n,\nu} + s_0^{n,*})(c_k^{n,\nu} + s_0^{n,*} + r^{n,\nu})} - \frac{\alpha_{0k}^n}{(d_k^{n,\nu} + s_0^{n,*})(d_k^{n,\nu} + s_0^{n,*} + r^{n,\nu})}\right)\right) \leq \tau, \tag{73}$$

$$\min_n\left(1 - \sum_{k=1}^{K}\left(\frac{\alpha_{0k}^n}{(c_k^{n,\nu} + s_0^{n,*})(c_k^{n,\nu} + s_0^{n,*} + r^{n,\nu})} - \frac{\alpha_{0k}^n}{(d_k^{n,\nu} + s_0^{n,*})(d_k^{n,\nu} + s_0^{n,*} + r^{n,\nu})}\right)\right) \geq -\tau. \tag{74}$$

We begin by considering the condition in (74). This condition can be written as

$$\max_n \sum_{k=1}^{K}\left(\frac{\alpha_{0k}^n}{(c_k^{n,\nu} + s_0^{n,*})(c_k^{n,\nu} + s_0^{n,*} + r^{n,\nu})} - \frac{\alpha_{0k}^n}{(d_k^{n,\nu} + s_0^{n,*})(d_k^{n,\nu} + s_0^{n,*} + r^{n,\nu})}\right) \leq \tau + 1. \tag{75}$$

Let χ_1 denote the term on the left hand side of (75). We first note that each term in the summand is a monotonically decreasing function of $r^{n,\nu}$. Since $s_0^{n,*} + r^{n,\nu} = s_0^{n,\nu} \geq 0$, χ_1 can be bounded as follows.

$$\begin{aligned}
\chi_1 &\leq \max_n \sum_{k=1}^{K}\left(\frac{\alpha_{0k}^n}{(c_k^{n,\nu} + s_0^{n,*})c_k^{n,\nu}} - \frac{\alpha_{0k}^n}{(d_k^{n,\nu} + s_0^{n,*})d_k^{n,\nu}}\right) \\
&= \max_n \sum_{k=1}^{K} \alpha_{0k}^n \frac{(d_k^{n,\nu})^2 - (c_k^{n,\nu})^2 + s_0^{n,*}(d_k^{n,\nu} - c_k^{n,\nu})}{(c_k^{n,\nu} + s_0^{n,*})c_k^{n,\nu}(d_k^{n,\nu} + s_0^{n,*})d_k^{n,\nu}} \\
&= \max_n \sum_{k=1}^{K} \frac{s_k^{n,\nu}(2c_k^{n,\nu} + s_0^{n,*} + \frac{s_k^{n,\nu}}{\alpha_{0k}^n})}{(c_k^{n,\nu} + s_0^{n,*})c_k^{n,\nu}(c_k^{n,\nu} + s_0^{n,*} + \frac{s_k^{n,\nu}}{\alpha_{0k}^n})(c_k^{n,\nu} + \frac{s_k^{n,\nu}}{\alpha_{0k}^n})}.
\end{aligned} \tag{76}$$

One can check that each term in the summand in (76) is a monotonically decreasing function of $s_k^{n,\nu}$. Hence, we have

$$\chi_1 \leq \max_n \sum_{k=1}^{K} \frac{S_{\max,k}^n(2c_k^{n,\nu} + s_0^{n,*} + \frac{S_{\max,k}^n}{\alpha_{0k}^n})}{(c_k^{n,\nu} + s_0^{n,*})c_k^{n,\nu}(c_k^{n,\nu} + s_0^{n,*} + \frac{S_{\max,k}^n}{\alpha_{0k}^n})(c_k^{n,\nu} + \frac{S_{\max,k}^n}{\alpha_{0k}^n})}. \tag{77}$$

Similarly, each term in the summand in (77) is a monotonically decreasing function of $c_k^{n,\nu}$. Hence,

$$\chi_1 \leq \max_n \sum_{k=1}^{K} \frac{S^n_{\max,k}(2c^n_{\min,k} + s_0^{n,*} + \frac{S^n_{\max,k}}{\alpha^n_{0k}})}{(c^n_{\min,k} + s_0^{n,*})c^n_{\min,k}(c^n_{\min,k} + s_0^{n,*} + \frac{S^n_{\max,k}}{\alpha^n_{0k}})(c^n_{\min,k} + \frac{S^n_{\max,k}}{\alpha^n_{0k}})}. \quad (78)$$

Finally, one can check that each term in the summand in (78) is a monotonically decreasing function of $s_0^{n,*}$. Therefore, we can write

$$\chi_1 \leq \max_n \sum_{k=1}^{K} \frac{S^n_{\max,k}(2c^n_{\min,k} + \frac{S^n_{\max,k}}{\alpha^n_{0k}})}{(c^n_{\min,k})^2(c^n_{\min,k} + \frac{S^n_{\max,k}}{\alpha^n_{0k}})^2}.$$

Therefore, a sufficient condition for (74) to be satisfied is

$$\max_n \sum_{k=1}^{K} \frac{S^n_{\max,k}(2c^n_{\min,k} + \frac{S^n_{\max,k}}{\alpha^n_{0k}})}{(c^n_{\min,k})^2(c^n_{\min,k} + \frac{S^n_{\max,k}}{\alpha^n_{0k}})^2} \leq \tau + 1. \quad (79)$$

We now proceed to provide a sufficient condition for (73) to be satisfied at all iterations. This condition can be written as

$$\chi_2 = \min_n \sum_{k=1}^{K} \left(\frac{\alpha^n_{0k}}{(c_k^{n,\nu} + s_0^{n,*})(c_k^{n,\nu} + s_0^{n,*} + r^{n,\nu})} - \frac{\alpha^n_{0k}}{(d_k^{n,\nu} + s_0^{n,*})(d_k^{n,\nu} + s_0^{n,*} + r^{n,\nu})} \right) \geq 1 - \tau. \quad (80)$$

Noting that each term in the summand is monotonically decreasing in $r^{n,\nu}$, we have

$$\begin{aligned}
\chi_2 &\geq \min_n \sum_{k=1}^{K} \left(\frac{\alpha^n_{0k}}{(c_k^{n,\nu} + s_0^{n,*})(c_k^{n,\nu} + S^n_{\max,0})} - \frac{\alpha^n_{0k}}{(d_k^{n,\nu} + s_0^{n,*})(d_k^{n,\nu} + S^n_{\max,0})} \right) \\
&= \min_n \sum_{k=1}^{K} \alpha^n_{0k} \frac{(c_k^{n,\nu} + s_0^{n,*})\left(\frac{s_k^{n,\nu}}{\alpha^n_{0k}}\right) + \frac{s_k^{n,\nu}}{\alpha^n_{0k}}(c_k^{n,\nu} + S^n_{\max,0} + \frac{s_k^{n,\nu}}{\alpha^n_{0k}})}{(c_k^{n,\nu} + s_0^{n,*})(c_k^{n,\nu} + S^n_{\max,0})(d_k^{n,\nu} + s_0^{n,*})(d_k^{n,\nu} + S^n_{\max,0})} \\
&= \min_n \sum_{k=1}^{K} \frac{s_k^{n,\nu}}{(c_k^{n,\nu} + S^n_{\max,0})(d_k^{n,\nu} + S^n_{\max,0})} \left(\frac{1}{c_k^{n,\nu} + s_0^{n,*}} + \frac{1}{d_k^{n,\nu} + s_0^{n,*}} \right) \quad (81) \\
&\geq \min_n \sum_{k=1}^{K} \frac{s_k^{n,\nu}}{(c_k^{n,\nu} + S^n_{\max,0})(d_k^{n,\nu} + S^n_{\max,0})} \left(\frac{1}{c_k^{n,\nu} + S^n_{\max,0}} + \frac{1}{d_k^{n,\nu} + S^n_{\max,0}} \right), \quad (82) \\
&\geq \min_n \sum_{k=1}^{K} \Bigg(\frac{(\alpha^n_{0k})^3 s_k^{n,\nu}}{\left(\sum_{j=1,\, j\neq k}^{K} \alpha^n_{jk} S^n_{\max,j} + \alpha^n_{0k} S^n_{\max,0} + \sigma^n_k\right)^2 \left(\sum_{j=1,\, j\neq k}^{K} \alpha^n_{jk} S^n_{\max,j} + s_k^{n,\nu} + \alpha^n_{0k} S^n_{\max,0} + \sigma^n_k\right)} \\
&\quad + \frac{(\alpha^n_{0k})^3 s_k^{n,\nu}}{\left(\sum_{j=1,\, j\neq k}^{K} \alpha^n_{jk} S^n_{\max,j} + \alpha^n_{0k} S^n_{\max,0} + \sigma^n_k\right) \left(\sum_{j=1,\, j\neq k}^{K} \alpha^n_{jk} S^n_{\max,j} + s_k^{n,\nu} + \alpha^n_{0k} S^n_{\max,0} + \sigma^n_k\right)^2} \Bigg), \quad (83)
\end{aligned}$$

where (82) follows from observing that each term in the summand in (81) is monotonically decreasing in $s_0^{n,*}$. Since (83) is a monotone increasing function of $s_k^{n,\nu}$, we can use the lower bound $\eta^n_k \leq s_k^{n,\nu}$ (cf. (101)) to write

$$\chi_2 \geq \min_n \sum_{k=1}^{K} \Bigg(\frac{(\alpha^n_{0k})^3 \eta^n_k}{\left(\sum_{j=1,\, j\neq k}^{K} \alpha^n_{jk} S^n_{\max,j} + \alpha^n_{0k} S^n_{\max,0} + \sigma^n_k\right)^2 \left(\sum_{j=1,\, j\neq k}^{K} \alpha^n_{jk} S^n_{\max,j} + \eta^n_k + \alpha^n_{0k} S^n_{\max,0} + \sigma^n_k\right)}$$

$$+ \frac{(\alpha_{0k}^n)^3 \eta_k^n}{\left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)\left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \eta_k^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)^2}\Bigg). \tag{84}$$

Now, a sufficient condition for (80) to be satisfied is to have

$$\min_n \sum_{k=1}^{K} \Bigg(\frac{(\alpha_{0k}^n)^3 \eta_k^n}{\left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)^2\left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \eta_k^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)}$$
$$+ \frac{(\alpha_{0k}^n)^3 \eta_k^n}{\left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)\left(\sum_{j=1,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n + \eta_k^n + \alpha_{0k}^n S_{\max,0}^n + \sigma_k^n\right)^2}\Bigg) \geq 1 - \tau. \tag{85}$$

In summary, if conditions (72), (79), and (85) are simultaneously satisfied, the GIWFA iterations are guaranteed to converge to a unique Nash equilibrium point for the non-cooperative game (7). This completes the proof of Theorem 1.

Appendix B

A lower bound on $s_k^{n,\nu}$

Denote the interference level observed by User $k \in \mathcal{K}$ on the n-th tone at the ν-th iteration by $I_k^{n,\nu}$, where

$$I_k^{n,\nu} = \sum_{j=1}^{k-1} \alpha_{jk}^n s_j^{n,\nu} + \sum_{j=k+1}^{K} \alpha_{jk}^n s_j^{n,\nu-1} + \alpha_{0k}^n s_0^{n,\nu-1}. \tag{86}$$

Since

$$s_k^{n,\nu} \leq S_{\max,k}^n, \qquad \forall n \in \mathcal{N} \tag{87}$$

an upper bound on $I_k^{n,\nu}$ can be expressed as

$$I_k^{n,\nu} \leq I_{\max,k}^n = \sum_{j=0,\, j\neq k}^{K} \alpha_{jk}^n S_{\max,j}^n. \tag{88}$$

For every $k \in \mathcal{K}$, let the permutation $\pi_k(\cdot)$ be defined such that

$$\sigma_k^{\pi_k(1)} + I_{\max,k}^{\pi_k(1)} \leq \sigma_k^{\pi_k(2)} + I_{\max,k}^{\pi_k(2)} \leq \cdots \leq \sigma_k^{\pi_k(N)} + I_{\max,k}^{\pi_k(N)}. \tag{89}$$

Before we proceed with our analysis, we provide a brief discussion regarding the IWFA algorithm. User k's ν-th iteration of this algorithm is depicted in Figure 4. In this figure, we denote the water-level by μ_k^ν. Now, at each iteration, one can categorize the N tones into three classes; tones on which User k allocates power $S_{\max,k}^n$, tones on which User k performs standard water-filling, and tones on which User k puts no power. It is clear from Figure 4 that while the power allocated by User k on the first class of tones is not affected by the increase in water-level, if that exceeds a certain level, the power allocated on the

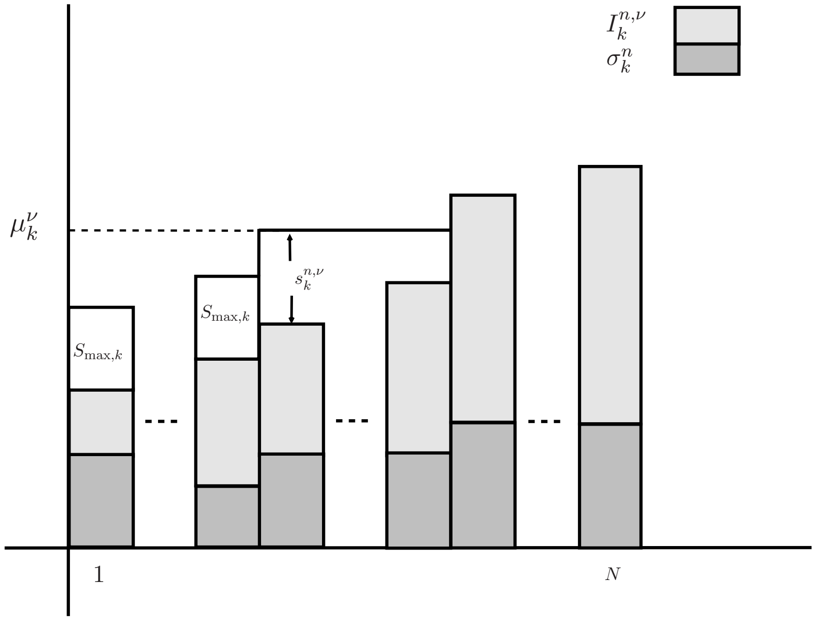


Fig. 4. At the ν-th iteration, User k uses power $s_k^{n,\nu}$ to water-fill over noise, σ_k^n and interference $I_k^{n,\nu}$.

remaining tones can only increase if μ_k^ν increases. Furthermore, we note that the constraints in (87) serve to increase to the water-level. In other words, if the constraints in (87) were not enforced, the water-level would decrease in order to bring the power level in the respective tones up to the water-level. Since in this section we are considering a lower bound on $s_k^{n,\nu}$, a worst-case scenario would be to assume that none of the constraints in (87) is active. In this case we have

$$s_k^{\pi_k(n),\nu} = [\mu_k^\nu - (I_k^{\pi_k(n),\nu} + \sigma_k^{\pi_k(n)})]^+, \qquad \forall n \in \mathcal{N}, \tag{90}$$

where $[\cdot]^+$ denotes the projection onto the non-negative real line.

Assuming, for simplicity of exposition, that at the ν-th iteration the noise plus interference assumes distinct values on each tone, it is possible to identify N water-level intervals. In particular, the water-level within a certain interval would only cover a certain subset of tones. Let the number of tones covered by water at the ν-th iteration be m_k^ν and let these tones be denoted by $\hat{\pi}_k(1), \ldots, \hat{\pi}_k(m_k^\nu)$, where, unlike (89), $\hat{\pi}_k(\cdot)$ is an iteration-dependent permutation of tones such that

$$\sigma_k^{\hat{\pi}_k(1)} + I_k^{\hat{\pi}_k(1),\nu} \leq \sigma_k^{\hat{\pi}_k(2)} + I_k^{\hat{\pi}_k(2),\nu} \leq \cdots \leq \sigma_k^{\hat{\pi}_k(N)} + I_k^{\hat{\pi}_k(N),\nu}. \tag{91}$$

Our goal is to find a lower bound on m_k^ν, and to identify the tones that User $k \in \mathcal{K}$ is guaranteed to activate at every iteration of the GIWFA. For the tones $\hat{\pi}_k(1), \ldots, \hat{\pi}_k(m_k^\nu)$, the term inside the square brackets (90) is non-negative, and this term is strictly negative for all remaining tones. Using this notation, we can express the water level explicitly as

$$\mu_k^\nu = \frac{1}{m_k^\nu}\Big(P_k + \sum_{i=1}^{m_k^\nu}(I_k^{\hat{\pi}_k(i),\nu} + \sigma_k^{\hat{\pi}_k(i)})\Big). \tag{92}$$

Substituting from (92) into (90), and noting that the choice of m_k^ν is such the term inside the square brackets (90) is non-negative for all j for which

$$\pi_k(j) \in \{\hat{\pi}_k(1), \dots, \hat{\pi}_k(m_k^\nu)\}. \tag{93}$$

$$s_k^{\pi_k(j),\nu} = \frac{1}{m_k^\nu}\Big(P_k + \sum_{i=1}^{m_k^\nu}(I_k^{\hat{\pi}_k(i),\nu} + \sigma_k^{\hat{\pi}_k(i)})\Big) - (I_k^{\pi_k(j),\nu} + \sigma_k^{\pi_k(j)}). \qquad \forall j \text{ for which (93) holds}, \tag{94}$$

Observe that if for the $\pi_k(j)$-th tone (93) does not hold, then the definition of m_k^ν implies that $s_k^{\pi_k(j),\nu} = 0$, and this tone is not used by User k at the νth iteration, and hence is not in the set of interest.

Let $m_k \in \{1, \dots, N\}$ be the desired lower bound on m_k^ν. Furthermore, let $\sigma_k^{(i)}$ denote the noise variance of User $k \in \mathcal{K}$ that satisfies $\sigma_k^{(i)} \leq \sigma_k^{(i+1)}$ for $i = 1, \dots, N-1$. We will show that if m_k is defined to be the largest integer for which

$$(m_k - 1)(\sigma_k^{\pi_k(j)} + I_{\max,k}^{\pi_k(j)}) \leq P_k + \sum_{i=1}^{m_k-1} \sigma_k^{(i)}, \tag{95}$$

is satisfied for all $j \leq m_k$, then $m_k \leq m_k^\nu$, $\forall \nu$. Since m_k satisfies (95), then m_k also satisfies

$$(m_k - 1)(\sigma_k^{\pi_k(j)} + I_k^{\pi_k(j),\nu}) \leq P_k + \sum_{\substack{i=1 \\ \hat{\pi}_k(i) \neq \pi_k(j)}}^{m_k} (\sigma_k^{\hat{\pi}_k(i)} + I_k^{\hat{\pi}_k(i),\nu}), \tag{96}$$

where $\hat{\pi}_k(\cdot)$ is the permutation of tones defined in (91). This is because the right hand side of (96) is at least as great as the right hand side of (95) and the left hand side is less than or equal to the left hand side of (95).

Now, (96) is equivalent to writing

$$\frac{1}{m_k}\Big(P_k + \sum_{i=1}^{m_k}(I_k^{\hat{\pi}_k(i),\nu} + \sigma_k^{\hat{\pi}_k(i)})\Big) - (I_k^{\pi_k(j),\nu} + \sigma_k^{\pi_k(j)}) \geq 0. \tag{97}$$

We now compare (97) with (94). Since by definition, m_k^ν is the largest integer for which the right hand side of (94) is greater than or equal to zero, we conclude that m_k is less than or equal to m_k^ν. However, from (95), we note that the definition of m_k does not depend on the iterations. Hence, from (95), we know that the tones $\pi_k(1), \dots, \pi_k(m_k)$ are going to be activated by User k in each iteration.

Using the fact that m_k is a lower bound on the number of tones that are going to be activated, we can write a lower bound on the water level, μ_k^ν at the ν-th iteration. In particular, using (92) and (93), it is easy to see that

$$\mu_k^\nu \geq \frac{1}{N}\big(P_k + \sum_{i=1}^{m_k}(\sigma_k^{\pi_k(i)} + I_k^{\pi_k(i),\nu})\big). \tag{98}$$

Now, substituting from (98) into (90), we have

$$s_k^{\pi_k(n),\nu} \geq \Big[\frac{1}{N}\big(P_k + \sum_{i=1}^{m_k}\sigma_k^{\pi_k(i)}\big) - \Big(1 - \frac{1}{N}\Big)I_k^{\pi_k(n),\nu} - \sigma_k^{\pi_k(n)}\Big]^+, \qquad \forall n \in \mathcal{N}, \tag{99}$$

$$\geq \Big[\frac{1}{N}\big(P_k + \sum_{i=1}^{m_k}\sigma_k^{\pi_k(i)}\big) - \Big(1 - \frac{1}{N}\Big)I_{\max,k}^{\pi_k(n)} - \sigma_k^{\pi_k(n)}\Big]^+, \qquad \forall n \in \mathcal{N}, \tag{100}$$

$$= \Big[\frac{1}{N}\big(P_k + \sum_{i=1}^{m_k}\sigma_k^{\pi_k(i)}\big) - \Big(1 - \frac{1}{N}\Big)\sum_{j=0,\, j\neq k}^{K}\alpha_{jk}^{\pi_k(n)}S_{\max,j}^{\pi_k(n)} - \sigma_k^{\pi_k(n)}\Big]^+, \qquad \forall n \in \mathcal{N}. \tag{101}$$

Finally, we define η_k^n as

$$\eta_k^n \triangleq \Big[\frac{1}{N}\big(P_k + \sum_{i=1}^{m_k}\sigma_k^{\pi_k(i)}\big) + \big(\frac{1}{N} - 1\big)\sum_{j=0,\, j\neq k}^{K}\alpha_{jk}^{n}S_{\max,j}^{n} - \sigma_k^{n}\Big]^+, \tag{102}$$

where m_k is the largest integer for which (95) is satisfied, and the tone permutations $\pi_k(\cdot)$ are defined in (89) for all $k \in \mathcal{K}$. Hence, from (101) we have that η_k^n is an iteration-independent lower bound on $s_k^{n,\nu}$.

Appendix C

Proving the equivalence of (69) and (70)

In order to show that the condition in (69) is equivalent to that in (70), we notice that the 2×2 matrix on the right hand side of (69) is rank 1. Let us denote this matrix by Z; i.e.,

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & (1 - \|\Upsilon^\nu\|_2)^{-1}\|\sum_{k=1}^K F_k\|_2 \end{bmatrix} \begin{bmatrix} \|A^{-1}B\|_2 & \|A^{-1}\beta\| \\ \|A^{-1}B\|_2 & \|A^{-1}\beta\| \end{bmatrix}. \tag{103}$$

The condition in (69) is equivalent to $\|ZZ^T\|_2 < 1$. However, because Z is rank 1, then ZZ^T is also rank 1, and we have

$$\|ZZ^T\|_2 = \mathrm{Tr}(ZZ^T) = \Big(1 + (1 - \|\Upsilon^\nu\|_2)^{-2}\|\sum_{k=1}^K F_k\|_2^2\Big)(\|A^{-1}B\|_2^2 + \|A^{-1}\beta\|^2) < 1, \tag{104}$$

which is the condition given in (70).

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