Due Tuesday 09/24/13 (at the beginning of the class)

1. Problem 5.2 from the book (page 45; attached).
2. Problem 6.2 from the book (page 54; attached).
3. (a) Suppose that A and B are constant square matrices. Show that the state transition matrix for the time-varying system described by
$$\dot{x}(t) = e^{-At} B e^{At} x(t)$$
is
$$\Phi(t,s) = e^{-At} e^{(A+B)(t-s)} e^{As}\ .$$
(b) If A is an $n \times n$ matrix of full rank, show using the definition of the matrix exponential that
$$\int_0^t e^{A\sigma} d\sigma = [e^{At} - I]A^{-1}.$$
Using this result, obtain the solution to the linear time-invariant equation
$$\dot{x} = Ax + B\bar{u}\ , \quad x(0) = x_0$$
where $\bar{u}$ is a constant r-dimensional vector and B is an $(n \times r)$-dimensional matrix.
4. Consider the discrete-time system
$$\begin{aligned} x_{k+1} &= A\,x_k \ + \ B\,u_k \\ y_k &= C\,x_k \ + \ D\,u_k \\ x(k_0) &= x_0 \end{aligned}$$
with constant matrices A, B, C, and D.
   (a) Prove that this system is linear and time-invariant.
   (b) Using the definition of the $\mathcal{Z}$-transform prove that $\mathcal{Z}\left(A^k\right) = z\,R(z)$, where $R(z) := (z\,I - A)^{-1}$ is the resolvent of the matrix A.
   (c) For
$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0.3 \end{bmatrix}$$
determine $R(z)$. From the resulting expression for the resolvent compute the state transition matrix of the above system at $k = 9$.

P5.6 For every fixed $t_0 \geq 0$, the ith column of $\Phi(t, t_0)$ is the unique solution to

$$x(t+1) = A(t)x(t), \qquad x(t_0) = e_i, \qquad t \geq t_0,$$

where e_i is the ith vector of the canonical basis of $\mathbb{R}^n$.

This is just a restatement of Property P5.5 above.

P5.7 For every $t \geq s \geq \tau \geq 0$,

$$\Phi(t, s)\Phi(s, \tau) = \Phi(t, \tau).$$

Attention! The discrete-time state transition matrix $\Phi(t, t_0)$ may be singular. In fact, this will always be the case whenever one of $A(t-1), A(t-2), \ldots, A(t_0)$ is singular. □

Theorem 5.3 (Variation of constants). *The unique solution to*

$$x(t+1) = A(t)x(t) + B(t)u(t), \qquad y(t) = C(t)x(t) + D(t)u(t),$$

with $x(t_0) = x_0 \in \mathbb{R}^n$, $t \in \mathbb{N}$, *is given by*

$$x(t) = \Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)B(\tau)u(\tau), \qquad \forall t \geq t_0$$

$$y(t) = C(t)\Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} C(t)\Phi(t, \tau+1)B(\tau)u(\tau) + D(t)u(t), \quad \forall t \geq t_0$$

where $\Phi(t, t_0)$ *is the discrete-time state transition matrix.* □

5.4 EXERCISES

5.1 (Causality and linearity). Use equation (5.7) to show that the system

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x + D(t)u \tag{CLTV}$$

is causal and linear. □

5.2 (State transition matrix). Consider the system

$$\dot{x} = \begin{bmatrix} 0 & t \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ t \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \qquad x \in \mathbb{R}^2,\ u, y \in \mathbb{R}.$$

(a) Compute its state transition matrix

(b) Compute the system output to the constant input $u(t) = 1$, $\forall t \geq 0$ for an arbitrary initial condition $x(0) = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}'$. □

6.2 (Matrix powers and exponential). Compute A^t and e^{At} for the following matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \tag{6.7}$$