Lecture 8 9/26

Last time: - $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$

- $\phi(t; t_0) = e^{A(t-t_0)} = \phi(t - t_0)$

↳ for LTI system

- numerical computation of ϕ

Today: · Laplace Transform

- Impulse + freq. response
- transfer function
- examples!!!

Recall for LTI:

$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$y(t) = C \cdot e^{At} \cdot x_0 t + \int_0^t \underbrace{\left[C \cdot e^{A(t-\tau)} \cdot B \cdot D\delta(t-\tau)\right]}_{\text{transfer function}} u(\tau) d\tau \quad \ldots \boxed{*}$$

abstractly:

$$y(t) = C \cdot e^{At} \cdot x_0 + \int_0^t H(t-\tau) \cdot u(\tau) d\tau$$

Laplace Transforms

→ tool for dealing w/ LTI system in continuous time

(recall, in DT → Z-transform $F(z) = \mathcal{Z}\{f\} = \sum_{k=0}^{\infty} f_k \cdot z^{-k}$; $z \in \mathbb{C}$)

In CT : Laplace Transform

↳ $\boxed{F(s) = \mathcal{L}\{f\} = \int_0^{\infty} f(t) \cdot e^{-st} \, dt}$

*natural mode: if $x_{k+1} = a \cdot x_k \rightarrow x_k = a^k \cdot x_0$

$\dot{x}(t) = a\, x(t) \rightarrow x(t) = e^{at} \cdot x_0$

Properties: 1. Linearity

2. $\mathcal{L}\{dx(t)/dt\} = s X(s) - x(0)$

3.) for zero I.C. → $\mathcal{L}\{d^m x(t)/dt^m\} = s^m \cdot X(s)$

Most important property: 4. Convolution $\xrightarrow{\mathcal{L}}$ Multiplication

$$\mathcal{L}\left\{\int_0^t H(t-\tau)\,u(\tau)\,d\tau\right\} = H(s)\cdot U(s)$$

oh... and... 5. $\mathcal{L}\{\delta(t)\} = 1$

back to state model: (LTI)

$$\dot{x} = Ax + Bu \qquad \ldots (1)$$

$$y = Cx + Du \qquad \ldots (2)$$

$$x(0) = x_0$$

$A, B, C, D \rightarrow$ constant matrices

$$\mathcal{L}(1) \Rightarrow s\cdot X(s) - x_0 = A\cdot X(s) + B\cdot U(s)$$

$$\mathcal{L}(2) \Rightarrow Y(s) = CX(s) + DU(s)$$

$$X(s) = (s\cdot I - A)^{-1}x_0 + (sI-A)^{-1}\cdot B\cdot U(s)$$

$$Y(s) = C\cdot\underbrace{(sI-A)^{-1}}_{R(s)\text{ resolvent}}x_0 + [\underbrace{C(sI-A)^{-1}B + D}_{H(s)\text{ transfer}}]\,U(s) \ldots (*)$$

from $(*)$ + $\boxed{*}$ $\Rightarrow$

$$\mathcal{L}\{e^{At}\} = R(s) = (sI-A)^{-1}$$

$$\mathcal{L}\{H(t)\} = H(s) = C\cdot(sI-A)^{-1}\cdot B + D$$

*note: since $Y(s) = H(s)U(s)$ for $x_0 \equiv 0$

if $U(s) = 1 \rightarrow Y(s) = H(s)$

in time domain $u(t) = \delta(t) \rightarrow Y(t) = H(t) \rightarrow$ impulse response

Frequency Response: $H(s)\big|_{s=j\omega} = H(j\omega)$

the end. (for $\mathcal{L}$-transform)

ex// Double Integrator

$\ddot{y} = u$; y = position, u = force $\Big\}$ of a mass moving on a frictionless surface

state-space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u \qquad \begin{pmatrix} x_1 = y \\ x_2 = \dot{y} \end{pmatrix}$$

unforced w/o diff. eq.

$\ddot{y}(t) = 0 \rightarrow \dot{y}(t) = C_1 \rightarrow y(t) = C_1 t + C_2$

C_1 + C_2 → constant from I.C.

so set I.C!

$t = 0 \Rightarrow \dot{y}(0) = C_1$

$y(0) = C_1(0) + C_2$

$C_1 = \dot{y}(0)$: initial velocity

$C_2 = y(0)$: intitial condition

$$\int_0^t H(t-\tau)u(\tau)\,d\tau$$

$$\underbrace{H(t)}_{t} = \mathcal{L}^{-1}\{H(s) = \tfrac{1}{s^2}\}$$

$$\boxed{y(t) = \dot{y}(0)\,t + y(0)}$$

Alternative method to get here

$$\mathcal{L}\{\ddot{y} = u\} \Rightarrow s^2 \cdot Y(s) - s \cdot (y(0)) - \dot{y}(0) = U(s)$$

$$Y(s) = \frac{1}{s^2}\,\dot{y}(0) + \frac{1}{s} \cdot y(0) + \boxed{\frac{1}{s^2} \cdot U(s)}$$

hit w/ inverse $\mathcal{L}$, $\mathcal{L}^{-1}$

(step function)
heavy-side function $\longrightarrow \mathbb{1}(t)$

$$\boxed{\begin{aligned} y(t) = {} & (t \cdot \dot{y}(0) + 1 \cdot y(0)) \cdot \mathbb{1}(t) \\ & + \int_0^t (t - \tau) \cdot u(\tau)\,d\tau \end{aligned}}$$

↳ using $\mathcal{L}$-transform

ex// Matrix Exponential

$$e^{At} = \sum_{k=0}^{\infty} \frac{(A \cdot t)^k}{k!} = I + At + \frac{1}{2} A^2 t^2 + \dots$$

$$A \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In fact $A^k = 0$, $k = 2, 3, 4, \dots$

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_1(0) + t\,\dot{x}_2(0) \\ x_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} y(0) + t\,\dot{y}(0) \\ \dot{y}(0) \end{bmatrix}$$

Another Method: $e^{At} = \mathcal{L}^{-1}\{(SI - A)^{-1}\}$

$$SI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 0 & S \end{bmatrix}$$

$$(SI - A)^{-1} = 1/\det(SI - A) \cdot \operatorname{adj}(SI - A) = \frac{1}{S^2} \begin{bmatrix} S & 1 \\ 0 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{S} & \frac{1}{S^2} \\ 0 & 1/S^2 \end{bmatrix}$$

- Characteristic Polynomial: $f(S) = S^2 = 0 < \begin{matrix} S_1 = 0 \\ S_2 = 0 \end{matrix}$

$y(t) = c\, e^{S_1 \cdot t} + c_2\, e^{S_2 \cdot t}$ (holds if $S_1 \neq S_2$)

If $S_1 = S_2 \Rightarrow y(t) = C_1 e^{S_1 t} + C_2\, t\, e^{S_2 t}$

$$= C_1 + C_2 t$$

what is coming...

Phase Plane

$y(t) = \dot{y}(0)\,t + y(0)$
$\dot{y}(t) = \dot{y}(0)$