10/1 Lecture 9

Last Time: double-integrator example

$\ddot{y} = u \rightarrow$ input

$\ddot{y}$ ↳ output

Today: E-value decomposition

diagonalization of matrix (diagonal coord. form)

Unforced double-integrator

$\ddot{y} = 0 \;/\; \mathcal{L} \Rightarrow s^2 Y(s)$ - initial coordinate $= 0$

characteristic polynomial $f(s) = s^2$

roots determine types of solutions $\left.\begin{matrix} f(s) = 0 \\ s_1 = s_2 = 0 \end{matrix}\right.$

$y(t) = c_1 e^{0 \cdot t} + c_2 e^{0 \cdot t}(t) = c_1 + c_2 t$

note: a state space model given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

E-value decomposition of A:

$$A \cdot v = \lambda \cdot v \quad \ldots (1)$$

v ↳ vector

λ ↳ scalar

* interested in non-trivial solution ($v = 0$ always solves (1), but YAWN)

ex// $\dot{x} = Ax$

a.) $x(t) = e^{\lambda t} \cdot v$ ← assume this and ask under what conditions it solves this

$\cancel{\lambda e^{\lambda t} = e^{\lambda t}}$ $\lambda e^{\lambda t} v = e^{\lambda} \cdot A \cdot v$

$\lambda \cdot v = A \cdot v$

ex// continued

b.) hit $\dot{x} = Ax$ with Laplace Transform

$$(SI - A)X(S) = x(0)$$

back to E-value Decomposition of A:

rewrite (1) as: $(\lambda I - A)v = 0$ ... (2)

(if $(\lambda I - A)$ is invertible $\Rightarrow$ det $\neq 0$)

then $v = 0$ is a unique solution to (2) or (1)

If $\lambda \in \mathbb{C}$ is such that $\det(\lambda I - A) = 0$ than (1) has non-trivial solutions

ex// $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix}$$

$\Rightarrow \det(\lambda I - A) = \lambda \cdot \lambda - 0 \cdot (-1) = \lambda^2$

$\Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 0 \end{matrix}$

$\Rightarrow$ the only values for which (1) has non-trivial solution

(find E-value of A in hw)

ex// $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix}$$

$\Rightarrow \det(\lambda I - A) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$

$\Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -2 \end{matrix}$

find v_1 & v_2

$\rightarrow$ solve $(\lambda_i I - A)v_i = 0$

ex// continued

a.) $\lambda = -1$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} -a_1 - b_1 = 0 \\ 2a_1 + 2b_1 = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} b_1 = -a_1 \\ b_1 = -a_1 \end{matrix}$$

$$\Rightarrow v_1 = \begin{bmatrix} a_1 \\ -a_1 \end{bmatrix}$$

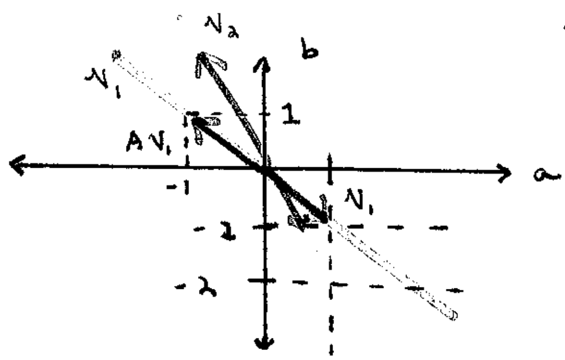
* in matlab : $v_1 = \frac{1}{\sqrt{a_1^2 + a_1^2}} \begin{bmatrix} a_1 \\ -a_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \text{sign}(a_1) \\ -\text{sign}(a_1) \end{bmatrix}$

b.) $\lambda_2 = -2$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} -2a_2 - b_2 = 0 \\ 2a_2 + b_2 = 0 \end{matrix} \quad \Rightarrow b_2 = -2a_2$$

$$v_2 = \begin{bmatrix} a_2 \\ -2a_2 \end{bmatrix} \Rightarrow v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} \text{sign}(a_2) \\ -2(\text{sign}(a_2)) \end{bmatrix}$$



$\mathbb{R}^2$

v_1 + v_2 determine direction of special significance

$A \cdot v_i = \lambda_i v_i$

if we act on e-vectors w/ A, we'll stay in the same direction (i.e. obtain $\lambda_i v_i$); interp for $\lambda_i \in \mathbb{C}$ later

let us assume that $A \in \mathbb{R}^{n \times n}$ has a full set of linearly independent vectors.

span $\{v_1, \ldots, v_n\} = \mathbb{R}^n$ (i.e any vector $x \in \mathbb{R}^n$ can be written as $x = \alpha_1 v_1 + \alpha_2 v_2 + \ldots$)

then...

then...

$$A \cdot v_i = \lambda_i v_i$$

$$[A \cdot v_1 + \dots A v_n] = [v_1 \lambda_1 + \dots + v_n \lambda_n] \in \mathbb{C}^{n \times n}$$

$$\begin{bmatrix} & A & \end{bmatrix} \underbrace{\begin{bmatrix} v_1, \dots v_n \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} v_1, \dots v_n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$A \cdot V = V \Lambda$$

since $\{v_1, \dots, v_n\}$ are linearly independent $\det V \neq 0$

$\Rightarrow$

$$\boxed{A = V \Lambda V^{-1} \Leftrightarrow \Lambda = V^{-1} A \cdot V}$$

~~[illegible]~~

Examples of diagonalizable matrices

1.) symmetric $(A = A^T)$

2.) normal matrices $A \cdot A^T = A^T \cdot A$

↳ "unitarily diagonalizable"

↳ ($v_1, \dots v_n$ are mutually orthogonal)

↳ $v_i \cdot v_j = 1 \rightarrow i = j$ or $0 \rightarrow i \neq j$

in this case, $V^{-1} = V^*$

Back to systems theory:

$$\dot{x} = Ax + Bu \quad \dots \quad (1)$$

$$y = Cx + Du \quad \dots \quad (2)$$

↳ introduce change of coordinates:

$$z := T^{-1} x \quad \Leftrightarrow \quad x = T \cdot z \quad \dots \quad (3)$$

$T \in \mathbb{R}^{n \times n}$ is invertable matrix

$$(3) \rightarrow (1) \Rightarrow \dot{x} = T \cdot \dot{z} = A \cdot T \cdot z + Bu$$

$$\dot{z} = T^{-1} A T \cdot z + T^{-1} B \cdot u$$

$$(3) \rightarrow (2) \Rightarrow y = C \cdot T \cdot z + D u$$

then $\dot{z} = \bar{A} z + \bar{B} u$ (1') where $\bar{A} = T^{-1} A T$

$y = \bar{C} z + \bar{D} u$ (2') $\qquad \bar{B} = T^{-1} B$

$\bar{C} = C \cdot T$

$\bar{D} = D$

A look @ transfer coordinates

for system (1'), (2')

$$G(S) = \bar{C}\cdot(SI-\bar{A})^{-1}\cdot\bar{B}+\bar{D}$$
$$= C\cdot T(S\underbrace{I}_{(T^{-1}\cdot T)} - T^{-1}AT)^{-1}\cdot T^{-1}\cdot B + D$$
$$= CT(T^{-1}(SI-A)\cdot T)^{-1}\cdot T\cdot B + D$$

(note $(PQR)^{-1} = R^{-1}Q^{-1}P^{-1}$)

$$= C\cdot T\cdot T^{-1}(SI-A)^{-1}T\cdot T^{-1}B + D$$
$$= C\cdot(SI-A)^{-1}B + D = H(S) \quad \text{Q.E.D} \quad \text{yay :)}$$

if A is diagonalizable w/ matrix of e-values $V = [v_1, \ldots, v_n]$
if we chose $T = V$

$$\Rightarrow \bar{A} = T^{-1}AT$$
$$= V^{-1}AV$$
$$= \Lambda$$

in z coordinates: $\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$

$$z_i(t) = e^{\lambda_i t} z_i(0)$$
$$z(t) = e^{\Lambda t}\cdot z(0)$$
$$x(t) = V\cdot e^{\Lambda t}\cdot V^{-1}x(0)$$
$$= e^{At}x(0)$$

* change of coordinates very important

* no class 10/3 (Thursday)