Lecture 11 10/10

last time: modal decomp of LTI systems
normal vs. non-normal matrices

today: recap of last time
modal stability

E-value decomposition:

$A v_i = \lambda_i v_i$

$v_i \to$ e-vector

$\lambda_i \to$ e-value

A: diagonalizable

$A = V \cdot \Lambda \cdot V^{-1}$

if $AA^T = A^+ A$ (A: normal)

e-vectors of A are mutually orthogonal

$\Rightarrow$ A: unitarily diagonalizable

$\Rightarrow A = V \Lambda V^*$

$\Lambda \to$ diagonal matrix $\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$

$$A \cdot f = V \Lambda V_f^* = \left(\sum_{i=1}^{n} \lambda_i \begin{bmatrix} v_i \end{bmatrix} \begin{bmatrix} \quad v_i^* \quad \end{bmatrix} \right) \cdot \begin{bmatrix} f \end{bmatrix}$$

$\hookrightarrow f \in \mathbb{R}^n$

$$v_i^* \cdot v_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$\to$ sum of rank 1 matrices

$$= \sum_{i=1}^{n} \lambda_i \underbrace{v_i^* \cdot f}_{\tau_i} \begin{bmatrix} v_i \end{bmatrix}$$

$\hookrightarrow$ scalar

Thus for normal A: no intersection between different eigen directions

For $\dot{x}(t) = A x(t) \to \begin{bmatrix} x(t) \end{bmatrix} = \sum e^{\lambda_i t} \begin{bmatrix} \quad v_i^* \quad \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix} \begin{bmatrix} v_i \end{bmatrix}$

$e^{\lambda_i t} \to$ advancement in time

If A is nonnormal, $(AA^T \neq A^TA)$ then $V^{-1} \neq V^*$

call it w^* $\rightarrow V^{-1} = W^*$

i.e. A is NOT unitarily diagonalizable
eigen-directions are NOT mutually orthogonal

$$Af = V \Lambda W^*_f = \sum_{i=1}^{n} \lambda_i \begin{bmatrix} v_i \end{bmatrix} \begin{bmatrix} \quad w_i^* \quad \end{bmatrix} \cdot \begin{bmatrix} f \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i w_i^* \cdot f \begin{bmatrix} v_i \end{bmatrix}$$ → not orthogonal

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} \cdot \begin{bmatrix} \quad w_i^* \quad \end{bmatrix} \begin{bmatrix} x_0 \end{bmatrix} \begin{bmatrix} v_i \end{bmatrix}$$

ex// $A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$ $AA^T \neq A^TA$ if $k \neq 0$

v_1 + v_2 nearly parallel for large k
$w_i^* A = \lambda_i w_i^*$ (left e-vector)
↳ we can always select them s.t.

$$w_i^* v_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

... Back to where we stopped last time

if there are complex e-values of A and A is diagonalizable then we can bring it to a block diagonal form:

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & \Lambda_{r+1} & & \\ & & & & \ddots & \\ & & & & & \Lambda_m \end{bmatrix}$$ $\{\lambda_1, \ldots \lambda_r\}$: real e-values of A

$\Lambda_i \in \mathbb{R}^2 \times \mathbb{R}^2$ correspond to complex e-values
$\lambda_i = \sigma_i + j\omega_i$

$$\Lambda_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}$$ * easy to show that e-value of Λ_i are $\sigma_i \pm j\omega_i$

If $\lambda_i \in \mathbb{C}$

$c_i \cdot e^{\lambda_i t} \cdot v_i = z(t)$

$\Rightarrow$ satisfies $\dot{z}(t) = A \cdot z(t)$

since $A \in \mathbb{R}^{n \times n}$, $Re(z(t))$ also satisfies $\frac{d(Re(z(t)))}{dt} = A \cdot Re(z(t))$

$\Rightarrow \cancel{Re}(z(t)) = [Re(c_i) + j\, Im(c_i)] \cdot e^{\sigma_i t} \cdot e^{j w_i t} \cdot [Re(v_i) + j\, Im(v_i)]$

$\cdot (\cos(w_i t) + j \sin(w_i t))$

$= Re(z(t)) + Im(z(t))$

~~Exercise~~

Exercise: write $Re(z(t)) = [Re(v_i) \quad Im(v_i)] \underbrace{\overbrace{\begin{bmatrix} & \\ & \end{bmatrix}}^{\text{time depend}}}_{} \underbrace{\begin{bmatrix} Re(c) \\ Im(c) \end{bmatrix}}_{\text{I.C. dependence}}$

thus: "plane" spanned by $Re(v_i)$ + $Im(v_i)$ invariant under system dynamics.

Question: ~~[illegible]~~ Can we diagonalize every matrix?

ex// $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \lambda \in \mathbb{R}$

E-values $\lambda_1 = \lambda_2 = \lambda$

(algebraic multiplicity 2)

$\lambda I - A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

single e-vector

$v = \begin{bmatrix} * \\ 0 \end{bmatrix} \Rightarrow V = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$

Fact: Any matrix can be brought $\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{bmatrix}$ (*)

where J_i are given by:

a.) $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}$ for $\lambda_1 \ldots \lambda_k$ real + have linearly independent e-vector

b.) $\begin{bmatrix} \sigma_i & w_i \\ -w_i & \sigma_i \end{bmatrix}$ for complex λ_i

c) $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ for λ's w/ linearly dependent e-vector

In several: $\begin{bmatrix} \lambda^1 & & 0 \\ & \ddots & \\ 0 & & \lambda^1 \end{bmatrix}$

note: matrix ~~of~~ exp. of $(*)$: $\begin{bmatrix} e^{J_1 t} & & \\ & e^{J_2 t} & \\ & & \ddots \; e^{J_m t} \end{bmatrix}$

ex//

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow e^{Jt} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

ex//

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow e^{Jt} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

ex// Jordan: $\lambda_1 = -1$; $\lambda_2 = -3$

$\lambda_3 = 4 + 3j$

$\lambda_4 = 4 - 3j$

$\lambda_5 = \lambda_6 = \lambda_7 = 10$

(+1 linearly independent e-vector)

$$\begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix} = \text{blkdiag}(J_1, J_2, J_3)$$

$$J_1 = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \quad J_2 = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} \quad J_3 = \begin{bmatrix} 10 & 1 & 0 \\ 0 & 10 & 1 \\ 0 & 0 & 10 \end{bmatrix}$$

Summary: thus there is a change of coordinates to bring $\dot{x}(t) = Ax(t)$ into $\dot{z}(t) = \bar{A} \cdot z(t)$ where $\bar{A} = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}$

$\Rightarrow z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ $\quad \dot{z}_i(t) = J_i \cdot z_i(t)$

cannot do this w/ time-variant system