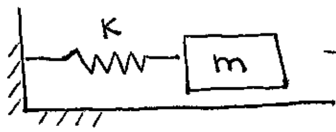
today: Lyapunov functions for LTI systems

ex// 

$\rightarrow y \qquad m\ddot{y} + ky = 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

energy: $E(y, \dot{y}) = \underbrace{\tfrac{1}{2} k y^2}_{\text{potential}} + \underbrace{\tfrac{1}{2} m \dot{y}^2}_{\text{kinetic}}$

equivalently: $E(x_1, x_2) = \tfrac{1}{2} k x_1^2(t) + \tfrac{1}{2} m x_2^2(t)$

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial E}{\partial x_2} \cdot \dot{x}_2$$

$$= k \cdot x_1 \cdot \dot{x}_1 + m x_2 \cdot \dot{x}_2$$

we'll evaluate $\frac{\partial E}{\partial t}$ along solutions of (MS)

$$\frac{\partial E}{\partial t} = k x_1 x_2 + m x_2 \cdot (-k/m \; x_1)$$

$$= k x_1 x_2 - k x_1 x_2$$

$$= 0 \Rightarrow E(t) = \text{constant}$$

LTI Systems:

→ stability via Lyapunov functions

$\dot{x} = Ax$

A = constant $n \times n$ matrix

Quadratic form on x

$$V(x) := x^T \cdot P \cdot x = [x_1, x_2, \ldots x_n] \underbrace{\begin{bmatrix} & & \\ & P & \\ & & \end{bmatrix}}_{n \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

↳ for Mass-spring:

$$V(x) = E(x) = [x_1 \quad x_2] \begin{bmatrix} \tfrac{1}{2}k & 0 \\ 0 & \tfrac{1}{2}m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$V : \mathbb{R}^n \rightarrow \mathbb{R}$

Restrictions on P: 1. $P = P^T$ (symmetric)

2. $P > 0$ (positive ~~derivative~~ definite)

note: anti-symmetric part of matrix doesn't contribute to a quadratic form $x^T P x$

discussion of 1)

Fact: any matrix can be written as:

$$P = P_s + P_a$$
$$= \frac{1}{2}(P + P^T) + \frac{1}{2}(P - P^T)$$
$$x^T P x = x^T \cdot P_s x + \frac{1}{2}(x^T P x - x^T P^T x)$$
$$= x^T P_s x + \frac{1}{2}(y^T x - x^T y)$$
$$= x^T P x$$
$$= x^T P_s x$$

discussion of 2)

Translation of $P = P^T > 0$ ↳ overloaded notation

for all $x \neq 0$; $x^T P x > 0$

equivalent characterization:

all e-values of P are positive

$\lambda_i(P) > 0$ for all $i = 1, \dots, n$

Note: $P = V \Lambda V^T$ → plug into $x^T P x$

$$x^T P x = x^T V \Lambda V^T x = z^T \Lambda z = \sum_{i=1}^{n} \lambda_i z_i^2$$

Note: $\lambda_{min}(P)\, x^T x \leq x^T P x \leq \lambda_{max}(P)\, x^T x$

⇒ If $P > 0$ ⇒ $\|x\| = \sqrt{x^T x} \to \infty$

⇒ $V(x) \to \infty$ ($V(x)$ is radially unbounded)

Plan: Determine $\frac{dV}{dt}$ and Study sign along the solutions of $\dot{x} = Ax$

$$\frac{dV}{dt} = \dot{x}^T P \cdot x + x^T P \dot{x} = (Ax)^T P \cdot x + x^T \cdot P \cdot A$$
$$= x^T A^T P x + x^T P A x$$
$$= x^T \underbrace{(A^T P + PA)}_{\text{symmetrix matrix → call it } -Q} x$$
$$= -x^T Q x \quad \text{where } \underbrace{A^T P + PA = Q}_{\text{algebraic Lyapanov equation}}$$

algebraic Lyapanov equation

matrix valued

determines what V is doing

Lyapunov Equation (algebraic)

↳ can be used to study stability properties of A

For nonlinear systems:

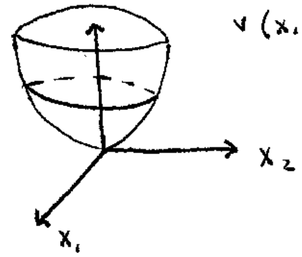
$V(x)$: positive-definite ⊕ radially unbounded

$\frac{dV}{dt}$: negative-definite (~~$\bar{x}=0$ is zero, negative otherwise~~)

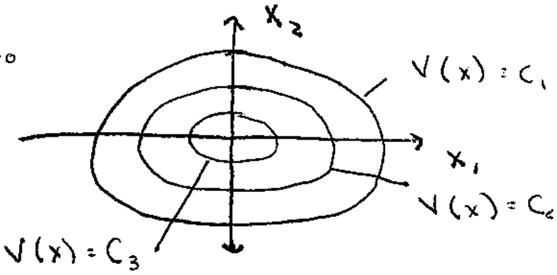
⇒ global asymptotic stability of $\bar{x} = 0$

$$\left\{\begin{array}{l} (x = 0 \text{ then } \left.\frac{dV}{dt}\right|_{x=0} = 0 \\ x \neq 0 \ \frac{dV}{dt} \text{ is negative}) \end{array}\right]$$

ex// $F = \frac{x^2}{x^2+1}$: not radially unbounded

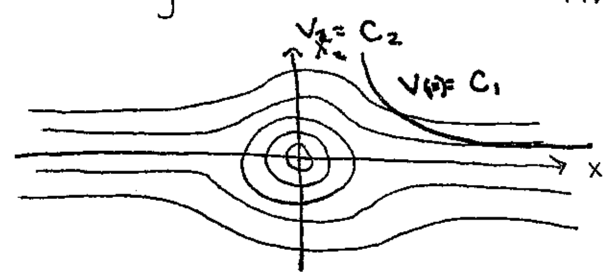


project onto x_1 x_2



$\frac{dV}{dt} < 0 \Rightarrow V(t)$ is a decreasing function of time

$$V(x_1, x_2) = \frac{x_1^2}{x_1^2+1} + x_2^2$$



Read Lyapunov Equation $A^T P + PA = -Q$

$$P = \int_0^\infty e^{A^T t} Q e^{At} \, dt$$

Exam next Thursday