Lecture 21 11/21 :)

Today: Kalman Rank Test is "on/off test"

Reachability gramian (K-step)

$$\boxed{P_K = R_K R_K^T = \sum_{i=0}^{K-1} A^i \cdot B \cdot B^T (A^T)^i}$$

K-step Reachability Gramian

$$A P_K A^T = \sum_{i=0}^{K-1} A^{i+1} B \cdot B^T (A^T)^{i+1}$$

$$j = i+1 \Rightarrow A P_K A^T = \sum_{j=1}^{K-1} A^j B B^T (A^T)^j + B B^T - B B^T$$

$$\Rightarrow A P_K A^T = P_{K+1} - B B^T$$

$$\hookrightarrow P_{K+1} = A P_K A^T + B \cdot B^T \quad * \text{ Lyapunov Eq}$$

for a given P_0, we can determine P_K for $K = 1, 2, \ldots$ from *

If $|\lambda_i(A)| < 1$ for $i = \{1, 2, \ldots n\}$

$$\lim_{K \to \infty} P_K = P_\infty$$

where P_∞ solves algebraic Lyapunov Eq.

$$A P_\infty A^T - P_\infty = - B \cdot B^T$$

(Matlab dlyap(A, BB'))

Properties of P_K:

1. $P_K = P_K^T$
2. $P_K \geq 0$ (positive semi-definite)

   $z^T \cdot P_K \cdot z = z^T R_K R_K^T z = y^T y \geq 0$
3. $P_\ell - P_K \geq 0 \quad \forall \ \ell \geq K$

   $P_{K+1} = \sum_{i=0}^{K} A^i B \cdot B^T (A^T)^i = \sum_{i=0}^{K-1} (\quad) + A^K B \cdot B^T (A^K)^T$

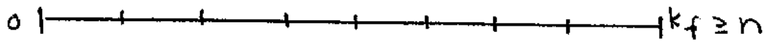
   $= P_K + A^K \cdot B \cdot B^T (A^K)^T$

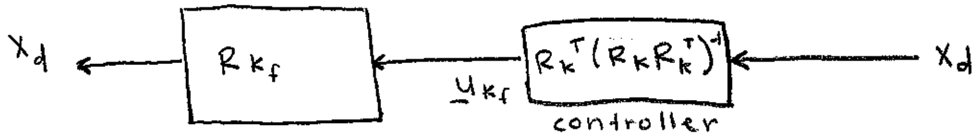
Now, Let our system be reachable

$\Rightarrow$ any $\begin{bmatrix} x_d \end{bmatrix} \in \mathbb{R}^n$ can be reached in n or fewer steps

$x(k+1) = Ax(k) + Bu(k)$

$x(0) = 0$



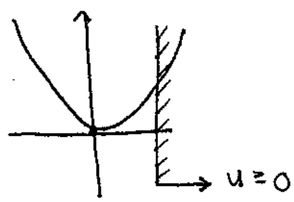


Pictorally: $\left[\, x_d \,\right]_{n\times 1} = \left[\quad \underset{\text{fat matrix}}{R_k} \quad\right] \left[\, \underline{u}_k \,\right]$ long vector

Minimize $\underline{u}_k^T \cdot \underline{u}_k$ s.t. $x_d - R_k \cdot \underline{u}_k = 0$

Problem data: x_d, R_k (given)

Unknown: (optimalization variable): $\underline{u}_k$



Lagrangian: $\mathcal{L}(\underline{u}_k, \lambda) = \underline{u}_k^T \cdot \underline{u}_k + \lambda^T (x_d - R_k \underline{u}_k)$

lagrang multiplier (price for violating constraint)

take variation wRt $\underline{u}_k$ + λ

$2 \cdot \underline{u}_k - R_k^T \lambda = 0$ (wrt $\underline{u}_k$)

$x_d - R_k \underline{u}_k = 0$ (wrt λ)

$\underline{u}_k = \frac{1}{2} R_k^T \cdot \lambda$

$x_d = \frac{1}{2}(R_k R_k^T)\lambda \Rightarrow \lambda = 2(R_k R_k^T)^{-1} \cdot x_d$

$$\Rightarrow \underline{u}_k^{opt} = \underbrace{R_k^T \cdot (R_k R_k^T)^{-1}}_{\text{pinv}(R_k) + R_k} \cdot x_d$$

energy of $\underline{u}_k^{opt}$:

$$(\underline{u}_k^{opt})^T \cdot \underline{u}_k^{opt} = x_d^T \cdot P_k^{-1} \cdot x_d$$

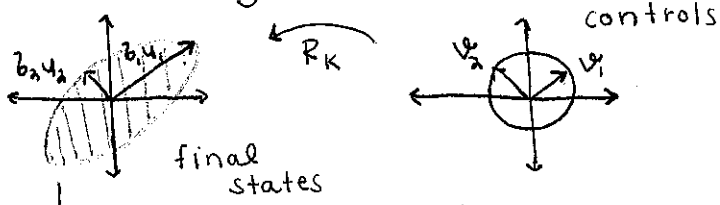
NOTE: $P_\ell \geq P_k$, $\ell \geq k$ $\Rightarrow$ it takes less + less energy to acheive our objective on longer time horizon

$P_\ell^{-1} \leq P_k^{-1}$, $\ell \geq k$

Note: $x_d = R_K \cdot \underline{u}_K = U \Sigma V^* \cdot \underline{u}_K = \sum_{i=1}^{r} \sigma_i \cdot u_i \cdot v_i^* \, \underline{u}_K$

if $\underline{u}_K = v_j \Rightarrow x_d = \sigma_j \cdot u_j$

Reachability elipsoid



↳ what we can reach

$R_K \cdot R_K^T = P_K = U \cdot \Sigma^2 \, U^* / U$

$P_K \cdot U = U \cdot \Sigma^2$

$P_K \cdot u_i = \sigma_i^2 u_i$

$\Rightarrow \boxed{P_K^{-1} \cdot u_i = \frac{1}{\sigma_i^2} \cdot u_i}$

large singular values = hard

if we want $x_d = u_i \Rightarrow x_d^T P_K^{-1} x_d = \frac{1}{\sigma^2}$

small σ_i's identify difficult-to-reach directions in $\mathbb{R}^n$