Optimal control – in 75 minutes or less ;-)

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Linear Quadratic Regulator (LQR)

- **Minimize quadratic objective subject to linear dynamic constraint**

minimize $\quad J(x,u) = \dfrac{1}{2}\displaystyle\int_0^T \big(\langle x(\tau), Q\,x(\tau)\rangle + \langle u(\tau), R\,u(\tau)\rangle \big)\,\mathrm{d}\tau \,+\, \dfrac{1}{2}\langle x(T), Q_T\,x(T)\rangle$

subject to $\quad A\,x(t) \,+\, B\,u(t) \,-\, \dot{x}(t) \,=\, 0$

$\qquad x(0) \,=\, x_0, \quad t \,\in\, [0,\, T]$

⋆ **state and control weights**

$$Q \,=\, Q^* \,\geq\, 0, \quad Q_T \,=\, Q_T^* \,\geq\, 0, \quad R \,=\, R^* \,>\, 0$$

$$\langle x(\tau), Q\,x(\tau)\rangle \;=\; x^*(\tau)\,Q\,x(\tau)$$

$$\langle u(\tau), R\,u(\tau)\rangle \;=\; u^*(\tau)\,R\,u(\tau)$$

minimize $\quad J(x,u) = \dfrac{1}{2}\displaystyle\int_0^T \big(\langle x(\tau), Q\, x(\tau)\rangle + \langle u(\tau), R\, u(\tau)\rangle \big)\, \mathrm{d}\tau \; + \; \dfrac{1}{2} \langle x(T), Q_T\, x(T)\rangle$

subject to $\quad A\, x(t) \; + \; B\, u(t) \; - \; \dot{x}(t) \; = \; 0$

$\quad x(0) \; = \; x_0, \;\; t \, \in \, [0, \, T]$

- **Features**
  - **optimization variable is a function**

$$u\colon \; [0, \, T] \; \longrightarrow \; \mathbb{R}^m$$

  - **state and control weights**

$$\begin{cases} Q, \, Q_T & \text{symmetric, positive semi-definite} \\ R & \text{symmetric, positive definite} \end{cases}$$

  - **infinite number of constraints**

• Introduce Lagrangian

$$\mathcal{L}(x, u, \lambda) \;=\; J(x, u) \;+\; \int_0^T \langle \lambda(\tau), A\, x(\tau) \;+\; B\, u(\tau) \;-\; \dot{x}(\tau) \rangle \, \mathrm{d}\tau$$

⋆ **form variations wrt x, u, λ**

$$\mathcal{L}(x, u + \tilde{u}, \lambda) \;-\; \mathcal{L}(x, u, \lambda) \;=\; \int_0^T \langle R\, u(\tau) \;+\; B^* \lambda(\tau), \tilde{u}(\tau) \rangle \, \mathrm{d}\tau \;=\; 0$$

$$\downarrow$$

$$u(t) \;=\; -R^{-1} B^* \lambda(t), \quad t \;\in\; [0,\, T]$$

necessary conditions for optimality:

wrt λ	$\Rightarrow$	$\dot{x}(t) \;=\; A\, x(t) \;+\; B\, u(t),$		$x(0) \;=\; x_0$
wrt x	$\Rightarrow$	$\dot{\lambda}(t) \;=\; -Q\, x(t) \;-\; A^* \lambda(t),$		$\lambda(T) \;=\; Q_T\, x(T)$
wrt u	$\Rightarrow$	$u(t) \;=\; -R^{-1} B^* \lambda(t),$		$t \;\in\; [0,\, T]$

Solution to finite horizon LQR

two-point boundary value problem:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -B\,R^{-1}B^* \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Q_T & -I \end{bmatrix} \begin{bmatrix} x(T) \\ \lambda(T) \end{bmatrix}$$

$$u(t) \;=\; -R^{-1}B^*\lambda(t)$$

- **Differential Riccati Equation**

can show: $\lambda(t) \;=\; X(t)\,x(t)$

$$-\dot{X}(t) \;=\; A^*\,X(t) \;+\; X(t)\,A \;+\; Q \;-\; X(t)\,B\,R^{-1}B^*X(t)$$

$$X(T) \;=\; Q_T$$

⋆ **optimal controller: determined by state-feedback**

$$u(t) \;=\; -K(t)\,x(t)$$

$$K(t) \;=\; R^{-1}B^*X(t)$$

Infinite horizon LQR

$$\text{minimize} \quad J \;=\; \frac{1}{2}\int_0^\infty \big(\langle x(\tau), Q\,x(\tau)\rangle \,+\, \langle u(\tau), R\,u(\tau)\rangle \big)\,\mathrm{d}\tau$$

$$\text{subject to} \quad \dot{x}(t) \;=\; A\,x(t) \,+\, B\,u(t)$$

- **Optimal controller:** $\left\{ \begin{array}{rcl} u(t) & = & -K\,x(t) \\ K & = & R^{-1}B^{*}X \end{array} \right.$

$\star$ $X = X^*$ – **non-negative solution to Algebraic Riccati Equation (ARE)**

$$A^*X \,+\, X A \,+\, Q \,-\, X B R^{-1} B^* X \;=\; 0$$

$$\left.\begin{array}{ll} (A,B) & \text{stabilizable} \\ (A,Q) & \text{detectable} \end{array}\right\} \;\Rightarrow\; \text{stability of } \dot{x}(t) \;=\; (A \,-\, B\,K)\,x(t)$$

Scalar example

$$\dot{x} = a\,x + u$$

$$J = \frac{1}{2}\int_0^\infty \left(q\,x^2(\tau) + r\,u^2(\tau)\right) \mathrm{d}\tau$$

- **Optimal controller**

$$k_{\mathrm{lqr}} = a + \sqrt{a^2 + \frac{q}{r}} \;\Rightarrow\; x(t) = \exp\left(-\sqrt{a^2 + \frac{q}{r}}\,t\right) x(0)$$

tradeoff:

	large q/r	small q/r
convergence rate	fast ✓	slow
control effort	large	low ✓

State-feedback H_2 controller

$$\text{minimize} \quad \lim_{t \to \infty} \mathcal{E}\Big(\langle x(t), Q\, x(t) \rangle \;+\; \langle u(t), R\, u(t) \rangle \Big)$$

$$\text{subject to} \quad \dot{x}(t) \;=\; A\, x(t) \;+\; B_d\, d(t) \;+\; B_u\, u(t)$$

$$\mathcal{E}\left(d(t_1)\, d^*(t_2)\right) \;=\; W_d\, \delta(t_1 \,-\, t_2)$$

- **Minimum variance controller**

state-feedback controller:

$$\begin{aligned} u(t) &= -K\, x(t) \\ K &= R^{-1} B_u^*\, X \\ 0 &= A^*\, X \;+\; X\, A \;+\; Q \;-\; X\, B_u\, R^{-1} B_u^*\, X \end{aligned}$$

State estimation

state equation: $\dot{x}(t) = A\,x(t) + B_d\,d(t) + B_u\,u(t)$

measured output: $y(t) = C\,x(t) + n(t)$

$d(t)$ – process disturbance; $n(t)$ – measurement noise

- **Estimator (observer)**

  ⋆ **copy of the system** + **linear injection term**

$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + 0 \cdot d(t) + B_u\,u(t) + L\,(y(t) - \hat{y}(t))$$

$$\hat{y}(t) = C\,\hat{x}(t) + 0 \cdot n(t)$$

  ⋆ **estimation error:** $\tilde{x}(t) = x(t) - \hat{x}(t)$

$$\dot{\tilde{x}}(t) = (A - L\,C)\,\tilde{x}(t) + \begin{bmatrix} B_d & -L \end{bmatrix} \begin{bmatrix} d(t) \\ n(t) \end{bmatrix}$$

$$\tilde{y}(t) = C\,\tilde{x}(t) + n(t)$$

(A, C): detectable $\Rightarrow$ can design L to provide stability of the error dynamics

Kalman filter

$$\begin{aligned}\dot{x}(t) &= A\,x(t) + B_d\,d(t) + B_u\,u(t)\\ y(t) &= C\,x(t) + n(t)\end{aligned}$$

$$\mathcal{E}\left(d(t_1)\,d^*(t_2)\right) = W_d\,\delta(t_1 - t_2); \quad \mathcal{E}\left(n(t_1)\,n^*(t_2)\right) = W_n\,\delta(t_1 - t_2)$$

- **Kalman filter: optimal estimator**
  - ⋆ **minimizes steady-state variance of** $\tilde{x}(t) = x(t) - \hat{x}(t)$

Kalman gain:

$$\boxed{\begin{aligned} L &= Y\,C^*\,W_n^{-1}\\ 0 &= A\,Y + Y\,A^* + B_d\,W_d\,B_d^* - Y\,C^*\,W_n^{-1}\,C\,Y\end{aligned}}$$

Output-feedback H_2 controller

$$\text{minimize} \quad \lim_{t \to \infty} \mathcal{E}\big(\langle x(t), Q\, x(t) \rangle \,+\, \langle u(t), R\, u(t) \rangle \big)$$

$$\text{subject to} \quad \begin{aligned} \dot{x}(t) \;&=\; A\, x(t) \,+\, B_d\, d(t) \,+\, B_u\, u(t) \\ y(t) \;&=\; C\, x(t) \,+\, n(t) \end{aligned}$$

$$\mathcal{E}\left(d(t_1)\, d^*(t_2)\right) \;=\; W_d\, \delta(t_1 \,-\, t_2); \quad \mathcal{E}\left(n(t_1)\, n^*(t_2)\right) \;=\; W_n\, \delta(t_1 \,-\, t_2)$$

- **Minimum variance controller**

observer-based controller:

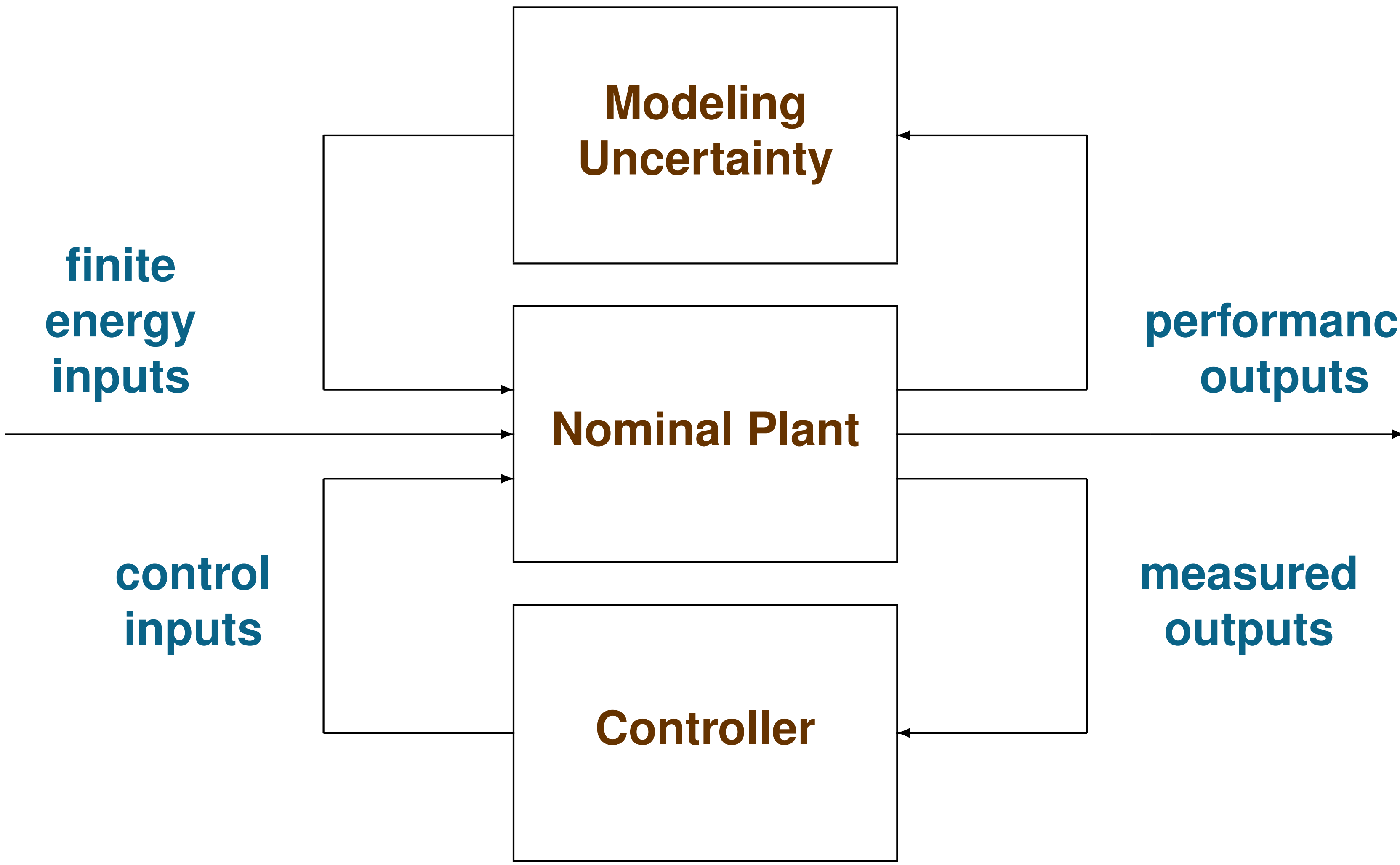
$$\begin{aligned} \dot{\hat{x}}(t) \;&=\; (A \,-\, L\, C)\, \hat{x}(t) \,+\, B_u\, u(t) \,+\, L\, y(t) \\ u(t) \;&=\; -K\, \hat{x}(t) \end{aligned}$$

⋆ **feedback and observer gains:**

$$\begin{cases} K & \text{LQR gain} \\ L & \text{Kalman gain} \end{cases}$$

H_∞ controller

- BLENDS CLASSICAL WITH OPTIMAL CONTROL



Take Pete's Robust Control course (EE/AEM 5235) next semester