Nonlinear Systems

Lecture 19 04/04/13

Last time:

→ Example of MRAC

$$\dot{y} = ay + u \qquad (1)$$

a: constant but unknown

MRAC: Adaptive controller that makes (1) behave like a reference model

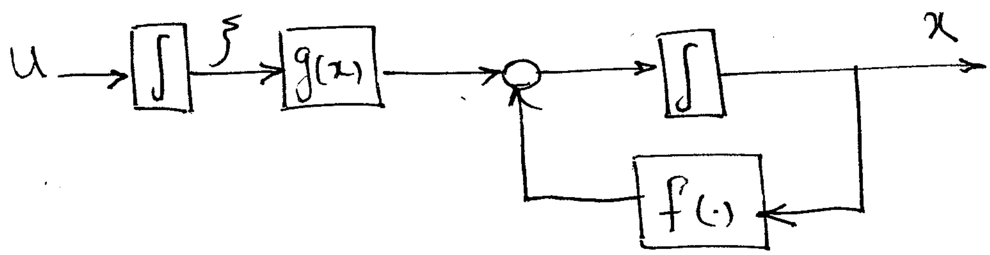
$$\dot{y}_m = -\underbrace{a_m}_{a_m > 0} y_m + \underbrace{r}_{\text{reference signal}}$$

→ Integrator backstepping

$$\dot{x} = f(x) + g(x)\xi \qquad (1) \quad (\text{affine dependence on } \xi)$$

$ax+b$ (affine) Vs ax (linear)

$$\dot{\xi} = u \qquad (2)$$



If ξ was control ($\xi = \alpha(x)$) and guarantees global asymptotic stability (GAS) of $\bar{x} = 0$ for (1) with a Lyapunov function $V(x)$

$$\frac{\partial V}{\partial x}\left[f(x) + g(x)\alpha(x)\right] < 0 \quad , \forall x \neq 0$$

then backstepping tells you how to design $u = \alpha_2(x, \xi)$ that provides GAS of $(\bar{x}, \bar{\xi}) = (0,0)$ of (1)-(2).

Ex

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = u$$

step 1 : Consider $\dot{x}_1 = x_1^2 + x_2$ and think of x_2 as a control.

Propose $V_1(x_1) = \frac{1}{2} x_1^2$

$$\dot{V}_1 = x_1 \dot{x}_1 = x_1(x_1^2 + x_2)$$

Eg. $x_2 = -x_1^2 - K_1 x_1 \qquad K_1 > 0$

or

$x_2 = -x_1^2 - K_1 x_1^3 \qquad K_1 > 0$

or

$\vdots$

Abstractly : $x_2 = \alpha(x_1)$

If x_2 was control $\Rightarrow$ done!

But it is not $\Rightarrow$ need to account for the fact that x_2 is a state variable (not control)

step 2: $z_2 = x_2 - \alpha(x_1)$

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}(x_1)$$

$$= u - \frac{\partial \alpha}{\partial x_1}\dot{x}_1 = u - \frac{\partial \alpha}{\partial x_1}[x_1^2 + x_2]$$

$$= u - \frac{\partial \alpha}{\partial x_1}[x_1^2 + z_2 + \alpha(x_1)]$$

So we introduced the new variable z_2, found an equation for it.

$\Rightarrow$ Augment Lyapunov function from step (1):

$$V_a(x_1, z_2) = V_1(x_1) + \tfrac{1}{2} z_2^2 = \frac{1}{2} x_1^2 + \tfrac{1}{2} z_2^2$$

$$\dot{V}_a = \dot{V}_1 + z_2 \dot{z}_2 = x_1 (x_1^2 + \alpha(x_1) + z_2) + z_2 (u - \dot{\alpha})$$

$$= \underbrace{x_1 (x_1^2 + \alpha(x_1))}_{-W_1(x_1)} + z_2 (u - \dot{\alpha} + x_1)$$

Choose:

$$\boxed{u = -x_1 + \dot{\alpha} - K_2 z_2, \quad K_2 > 0} \qquad (*)$$

to obtain: $\dot{V}_a = -W_1(x_1) - K_2 z_2^2 < 0$

GAS of the origin of (1)-(2).

Note! No reason to explicitly differentiate α in the expression for control $(*)$. Use instead the analytical expression for $\dot{\alpha}$:

$$\frac{\partial \alpha}{\partial t} = \frac{\partial \alpha}{\partial x_1} \dot{x}_1 = \frac{\partial \alpha}{\partial x_1} [x_1^2 + x_2]$$

Thus u given by $(*)$ is a static nonlinear state feedback control law, which provides GAS. $(u = \beta(x_1, x_2))$

Therefore backstepping provided a family of control laws in a constructive manner that achieve GAS.

Note! Backstepping yields:

→ A family of globally stabalizing control laws. These families are parameterized by different choices of stabilizing functions.

→ Control Lyapunov Function (CLF), which can be used to obtain additional control laws. (later)✓

Additional comments:

$$\begin{cases} \dot{x} = f(x) + g(x)\xi \\ \dot{\xi} = f_a(x,\xi) + g_a(x,\xi)\, u \end{cases}$$

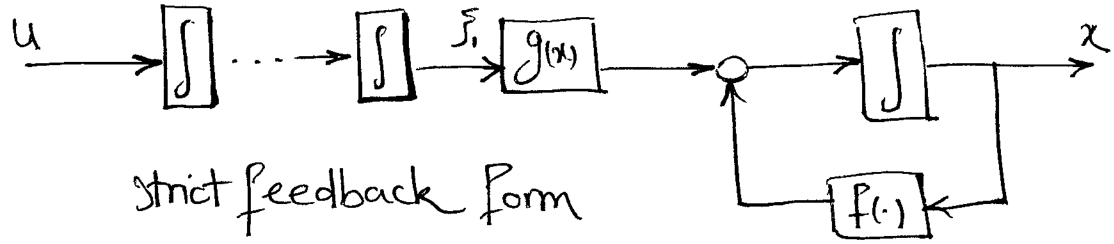
If $g_a(x,\xi) \neq 0$ for all x, ξ (same as before)

$$u = \frac{1}{g_a(x,\xi)} \left(\quad \cdots \quad \right)$$

Ex
$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u$$

Apply Backstepping.

$$z_3 = x_3 - \alpha_2(x_1, x_2)$$ and ... (same as before)



strict feedback form

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\,\underline{\underline{x_2}}$$
$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)\,\underline{\underline{x_3}}$$
$$\dot{x}_3 = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)\,\underline{\underline{u}}$$
$$\vdots$$

Ex $\dot{x}_1 = x_1^2 + x_2$

$\dot{x}_2 = u$

Objective : $x_1(t) \xrightarrow{t \to \infty} r(t)$

$y_r(t)$

$y_d(t)$

$$z_1 := x_1 - r$$

$$\dot{z}_1 = \dot{x}_1 - \dot{r} = x_1^2 + x_2 - \dot{r}$$

$$= (z_1 + r)^2 + x_2 - \dot{r}$$

$$\vdots$$

$$V_1(z_1) = \frac{1}{2} z_1^2$$

(Needs to know reference signal and its derivative)

Control Lyapunov Function (CLF)

$$\dot{x} = f(x) + g(x) \cdot u$$

$u = \alpha(x)$: Control law that gives

$$\underbrace{\frac{\partial V}{\partial x} f(x)}_{L_f V(x)} + \underbrace{\frac{\partial V}{\partial x} g(x)}_{L_g V(x)} \alpha(x) < 0$$

$$L_f V(x) + L_g V(x) . \alpha(x) < 0$$

If there are states x s.t. $L_gV(x) = 0$ and $L_fV(x) < 0$ at these points then control does not enter into equations for the derivative of V.

Propose $V(x)$:

$$\dot{V}\Big|_{(1)} = \frac{\partial V}{\partial x}[f(x) + g(x)u] = L_fV(x) + \underbrace{L_gV(x)}_{\text{when } = 0}.u$$

$$\Rightarrow L_fV(x) < 0$$

Def: A smooth positive definite radially unbounded function $V(x)$ is called a CLF for the system $\dot{x} = f(x) + g(x)u$ if for all $x \neq 0$ $L_gV(x) = 0 \Rightarrow L_fV(x) < 0$.

There are many ways to generate control laws from CLF:

E.g. Sontag's Formula (Eduardo Sontag, Rutgers, NJ)

$$\alpha(x) = \begin{cases} 0 & , \quad L_gV(x) = 0 \\ -\dfrac{L_fV(x) + \sqrt{(L_fV(x))^2 + (L_gV(x))^2}}{L_gV(x)} & \text{o.w.} \end{cases}$$

Comment !

This control law is continuous if the CLF satisfies the "small control property"

$\forall \varepsilon > 0$, $\exists\ \delta > 0$ st. for each $x \neq 0$ with $\|x\| \leq \delta$

$\Rightarrow$ we can find u with $\|u\| \leq \varepsilon$ st.

$$L_f V(x) + L_g V(x) u < 0$$