Due Tu 04/08/14 (at the beginning of the class)

1. Khalil, Problem 3.8 (attached).
2. Khalil, Problem 3.13 (attached). For $\begin{bmatrix} x_{10} & x_{20} \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, simulate sensitivity equations and plot the time dependence of the corresponding sensitivity functions.
3. Khalil, Problem 4.14 (attached).
4. What kind of equilibrium stability (stable (in the sense of Lyapunov), or AS, or GAS) if any, is exhibited by the state representation of
   (a) The $\frac{1}{s^2}$ plant with no input, i.e. $\dot{x}_1 = x_2$, $\dot{x}_2 = 0$.
   (b) The magnetically suspended ball: $\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \frac{-c}{m}\frac{\bar{u}^2}{x_1^2} + g \end{array}$ with $\bar{u} = \sqrt{\frac{mg}{c}Y} = \text{const.}$
5. The Morse oscillator is a model that is frequently used in chemistry to study reaction dynamics. The equations for an unforced Morse oscillator are given by

$$\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & -\mu(e^{-x_1} - e^{-2x_1}). \end{array}$$

   (a) Find the equilibrium points of the system.
   (b) Investigate their stability properties.
6. Consider the system:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\,g(k_1 x_1 + k_2 x_2), \qquad k_1,\, k_2 \,>\, 0, \end{array}$$

   where the nonlinearity $g(\cdot)$ is such that

$$g(y)\,y \,>\, 0, \quad \forall\, y \,\neq\, 0$$

$$\lim_{|y|\to\infty} \int_0^y g(\xi)\,\mathrm{d}\xi \;=\; +\infty$$

   (a) Using an appropriate Lyapunov function, show that the equilibrium $x = 0$ is globally asymptotically stable.

   (b) Show that the saturation function $\text{sat}(y) = \text{sign}(y)\min\{1, |y|\}$ satisfies the above assumptions for $g(\cdot)$. What is the exact form of your Lyapunov function for this saturation nonlinearity?

   (c) Parts (a) and (b) imply that a double integrator with a saturating actuator

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \text{sat}(u) \end{array}$$

   can be stabilized with the state-feedback controller $u = -k_1 x_1 - k_2 x_2$. Design k_1 and k_2 to place the eigenvalues of the linearization at $-1 \pm j$, and simulate the resulting closed-loop system both with, and without, saturation. Compare the resulting trajectories. (Please provide plots of $x_1(t)$ and $x_2(t)$ rather than phase portraits.)

3.6 Let $f(t, x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t, x)\| \leq k_1 + k_2\|x\|, \quad \forall\ (t, x) \in [t_0, \infty) \times R^n$$

(a) Show that the solution of (3.1) satisfies

$$\|x(t)\| \leq \|x_0\| \exp[k_2(t - t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t - t_0)] - 1\}$$

for all $t \geq t_0$ for which the solution exists.

(b) Can the solution have a finite escape time?

3.7 Let $g : R^n \to R^n$ be continuously differentiable for all $x \in R^n$ and define $f(x)$ by

$$f(x) = \frac{1}{1 + g^T(x)g(x)}g(x)$$

Show that $\dot{x} = f(x)$, with $x(0) = x_0$, has a unique solution defined for all $t \geq 0$.

3.8 Show that the state equation

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{2x_2}{1 + x_2^2}, \quad x_1(0) = a\\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1 + x_1^2}, \quad x_2(0) = b\end{aligned}$$

has a unique solution defined for all $t \geq 0$.

3.9 Suppose that the second-order system $\dot{x} = f(x)$, with a locally Lipschitz $f(x)$, has a limit cycle. Show that any solution that starts in the region enclosed by the limit cycle cannot have a finite escape time.

3.10 Derive the sensitivity equations for the tunnel-diode circuit of Example 2.1 as L and C vary from their nominal values.

3.11 Derive the sensitivity equations for the Van der Pol oscillator of Example 2.6 as ε varies from its nominal value. Use the state equation in the x-coordinates.

3.12 Repeat the previous exercise by using the state equation in the z-coordinates.

3.13 Derive the sensitivity equations for the system

$$\dot{x}_1 = \tan^{-1}(ax_1) - x_1x_2, \qquad \dot{x}_2 = bx_1^2 - cx_2$$

as the parameters a, b, c vary from their nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$.

4.6 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(x_1 + x_2) - h(x_1 + x_2)$$

where h is continuously differentiable and $zh(z) > 0$ for all $z \neq 0$. Using the variable gradient method, find a Lyapunov function that shows that the origin is globally asymptotically stable.

4.7 Consider the system $\dot{x} = -Q\phi(x)$, where Q is a symmetric positive definite matrix and $\phi(x)$ is a continuously differentiable function for which the ith component ϕ_i depends only on x_i, that is, $\phi_i(x) = \phi_i(x_i)$. Assume that $\phi_i(0) = 0$ and $y\phi_i(y) > 0$ in some neighborhood of $y = 0$, for all $1 \leq i \leq n$.

(a) Using the variable gradient method, find a Lyapunov function that shows that the origin is asymptotically stable.

(b) Under what conditions will it be globally asymptotically stable?

(c) Apply to the case

$$n = 2, \ \phi_1(x_1) = x_1 - x_1^2, \ \phi_2(x_2) = x_2 + x_2^3, \ Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

4.8 (**[72]**) Consider the second-order system

$$\dot{x}_1 = \frac{-6x_1}{u^2} + 2x_2, \qquad \dot{x}_2 = \frac{-2(x_1 + x_2)}{u^2}$$

where $u = 1 + x_1^2$. Let $V(x) = x_1^2/(1 + x_1^2) + x_2^2$.

(a) Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in R^2 - \{0\}$.

(b) Consider the hyperbola $x_2 = 2/(x_1 - \sqrt{2})$. Show, by investigating the vector field on the boundary of this hyperbola, that trajectories to the right of the branch in the first quadrant cannot cross that branch.

(c) Show that the origin is not globally asymptotically stable.

Hint: In part (b), show that $\dot{x}_2/\dot{x}_1 = -1/(1 + 2\sqrt{2}x_1 + 2x_1^2)$ on the hyperbola, and compare with the slope of the tangents to the hyperbola.

4.9 In checking radial unboundedness of a positive definite function $V(x)$, it may appear that it is sufficient to examine $V(x)$ as $\|x\| \to \infty$ along the principal axes. This is not true, as shown by the function

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

(a) Show that $V(x) \to \infty$ as $\|x\| \to \infty$ along the lines $x_1 = 0$ or $x_2 = 0$.

(b) Show that $V(x)$ is not radially unbounded.

4.10 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x) x \, d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x) P f(x)$ is positive definite for all $x \in R^n$ and radially unbounded.

(c) Show that the origin is globally asymptotically stable.

4.11 Using Theorem 4.3, prove Lyapunov's first instability theorem: For the system (4.1), if a continuously differentiable function $V_1(x)$ can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along the trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.12 Using Theorem 4.3, prove Lyapunov's second instability theorem: For the system (4.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\lambda > 0$ and $W(x) \geq 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.13 For each of the following systems, show that the origin is unstable:

$$(1) \qquad \dot{x}_1 = x_1^3 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

$$(2) \qquad \dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = x_1^6 - x_2^3$$

Hint: In part (2), show that $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

4.14 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(x_1)(x_1 + x_2)$$

where g is locally Lipschitz and $g(y) \geq 1$ for all $y \in R$. Verify that $V(x) = \int_0^{x_1} y g(y)\, dy + x_1 x_2 + x_2^2$ is positive definite for all $x \in R^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.