Lecture 1

Intro to Non-Linear Systems:

(what is the course about?)

EE / AEM 5231 (Linear Systems)

$$\boxed{\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)} \quad (1)$$

→ State Equation

$\dot{x}(t) := \dfrac{dx(t)}{dt}$

$u(t) \in \mathbb{R}^m$: input → means that $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$, $u_i(t) \in \mathbb{R}$

$x(t) \in \mathbb{R}^n$: state

In this course, we'll study dynamical systems of this form:

$$(2) \quad \boxed{\dot{x} = f(x, u, t)}$$, where $f(x,u,t)$ is in general a non-linear function of x and u!

Clearly, (1) is a special case of (2):

$$f(x,u,t) = A(t)\,x(t) + B(t)\,u(t)$$

→ "principle of superposition is not correct anymore!"

In Time Invariant Case:

matrices $A(t)$ and $B(t)$ are constant (they don't depend on time) → $\begin{cases} A(t) = A \in \mathbb{R}^{n\times n} \\ B(t) = B \in \mathbb{R}^{n\times m} \end{cases}$ → (1) simplifies into

$$\boxed{\dot{x}(t) = Ax(t) + Bu(t)} \quad (1')$$

In time invariant case (2) becomes:

$$\dot{x} = f(x, u) \quad (2')$$

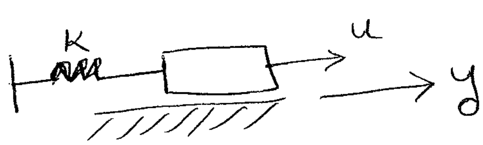
Solution to (1') given by:

$$x(t) = \underbrace{e^{At} x(0)}_{\text{initial conditions}} + \underbrace{\int_0^t e^{A(t-\tilde{\tau})} B \, u(\tilde{\tau}) \, d\tilde{\tau}}_{\text{inputs}}$$

In non-linear case we cannot separate these two influences!

So, No such expression for (2) or (2')!

<u>Example:</u>



Linear Spring: $m\ddot{y} + Ky = u$

Introducing $x_1 = y$: position, $x_2 = \dot{y}$: velocity $\rightarrow$ $\dot{x}_1 = \dot{y}$, $\dot{x}_2 = \ddot{y} = -\frac{K}{m} y + \frac{1}{m} u$

$$\rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{K}{m} x_1 + \frac{1}{m} u \end{cases} \longrightarrow$$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{m} & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

$\leadsto$ <u>Linear</u> & <u>Time Invariant</u>!

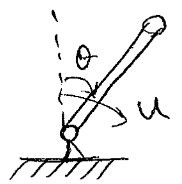
Example 2 :

$\dot{x} = \sin(x)$; $x(t) \in \mathbb{R}$ → Non linear & Time Invariant !

Eq. points of $\dot{x} = f(x)$ are Solutions to $\boxed{f(\bar{x}) = 0}$

(Constant Trajectories of our System) ⟹

Example 3 : Inverted pendulum



$\ddot{\theta} - \sin\theta = u$

$x_1 = \theta$

$x_2 = \dot{\theta}$

$$\rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) + u \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix}$$

⟹ Eq. points → points where unforced ($u \equiv 0$) system will stay, when it starts there !

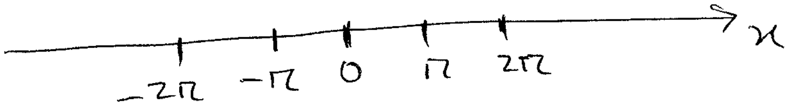
If $x(t) = \bar{x} = \text{Const}$ → $\boxed{\dot{x}(t) = \dot{\bar{x}} = 0 = f(\bar{x})}$

Ex 2:
$f(x) = \sin(x)$ ⟹ $\sin(\bar{x}) = 0$ → $\boxed{\bar{x} = k\pi \, , \; k = 0, \pm 1, \pm 2, \ldots}$

Ex 3: For I.P. : $\begin{bmatrix} \bar{x}_2 \\ \sin(\bar{x}_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ →

$$\boxed{(\bar{x}_1, \bar{x}_2) = (k\pi, 0) \; ; \quad k = 0, \pm 1, \pm 2, \ldots}$$

In Ex. 2 Eq. points given by:



"Can't happen in Nonlinear System"

Infinite number of $\boxed{\text{isolated eq. points}}$ (Same goes for I.P.)

Compare to : $\dot{x} = Ax$ $\rightarrow$ Eq. points : Solutions to $A\bar{x} = 0$

$$\rightarrow \underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} x \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{n\times 1}$$

If $\det A \neq 0 \Rightarrow \bar{x} = 0 \rightarrow$ unique eq. point!

If $\det A = 0 \Rightarrow \bar{x} \in \text{Null}(A)$: Subspace

Stability

Behavior of Eq. points in the presence of Small perturbations.

Given $\dot{x} = f(x)$ with $f(\bar{x}) = 0$ (i.e. $\bar{x} \rightarrow$ eq. point)

write : $\boxed{x(t) = \bar{x} + \tilde{x}(t)}$ $\rightarrow$ fluctuation around $\bar{x}$

$x(t)$: arbitrary trajectory

$\bar{x}$: eq. point

Taylor Series

$$\dot{\bar{x}} + \dot{\tilde{x}}(t) = f(\bar{x} + \tilde{x}(t)) = f(\bar{x}) + \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} (x - \bar{x}) + O(\|x-\bar{x}\|^2)$$

H.O.T

$$\dot{x} = f(\bar{x}) \longrightarrow \dot{\tilde{x}}(t) = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} (x-\bar{x})$$

$$\rightarrow \boxed{\dot{\tilde{x}}(t) = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}}}_{\text{Jacobian}} \tilde{x}(t)}$$