Lecture 04

Last time

- Limit cycles
  (EX: van der Pol oscillator)

- chaos
  (EX: Lorentz system)

- Fold Bifurcation

$\dot{x} = \alpha \pm x^2$

AS $\alpha \uparrow$ eq. point $\left\langle \begin{array}{l} \text{emerges} \quad (\dot{x} = \alpha - x^2) \\ \text{or} \\ \text{disappears} \quad (\dot{x} = \alpha + x^2) \end{array} \right.$

Today

- Transcritical $\qquad \dot{x} = \alpha x \pm x^2$
- pitchfork $\qquad \dot{x} = \alpha x \pm x^3$
- phase portraits of 2nd order systems

Transcritical:

$\dot{x} = \alpha x - x^2 = x(\alpha - x)$

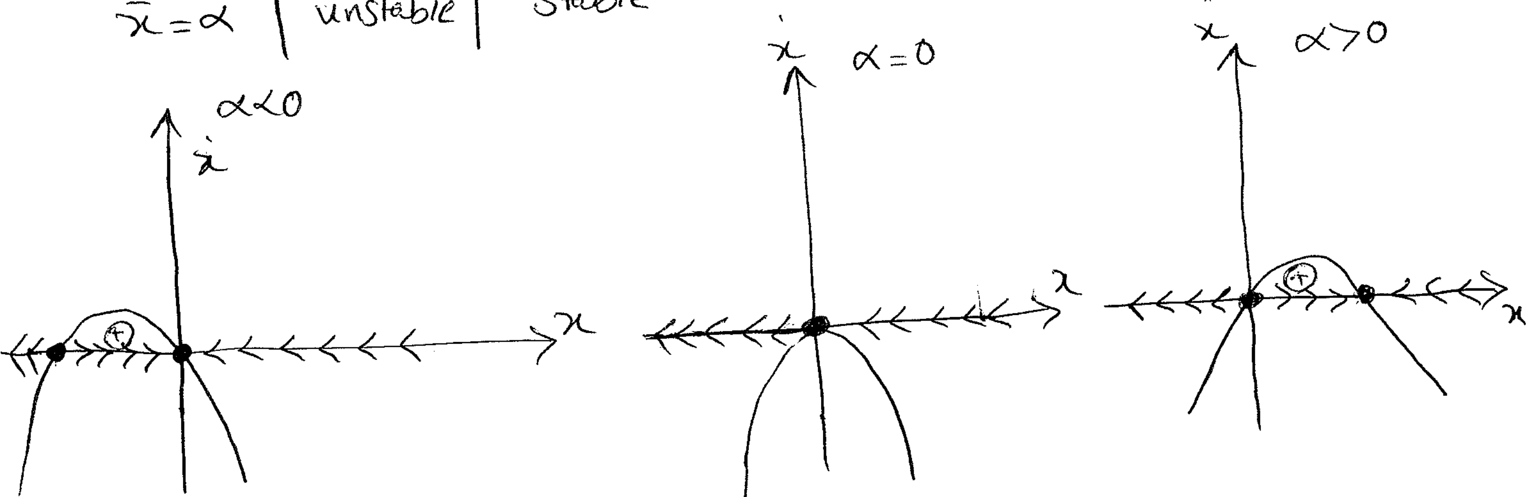
eq. points $\rightarrow 0 = \bar{x}(\alpha - \bar{x}) \Rightarrow \left\langle \begin{array}{l} \bar{x} = 0 \\ \bar{x} = \alpha \end{array} \right.$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \alpha - 2\bar{x} = \begin{cases} \alpha , & \bar{x} = 0 \\ -\alpha , & \bar{x} = \alpha \end{cases}$$

→ when $\alpha \neq 0$, one of them is stable and the other one is not. when $\alpha = 0$ we cannot decide about stability with this tool!

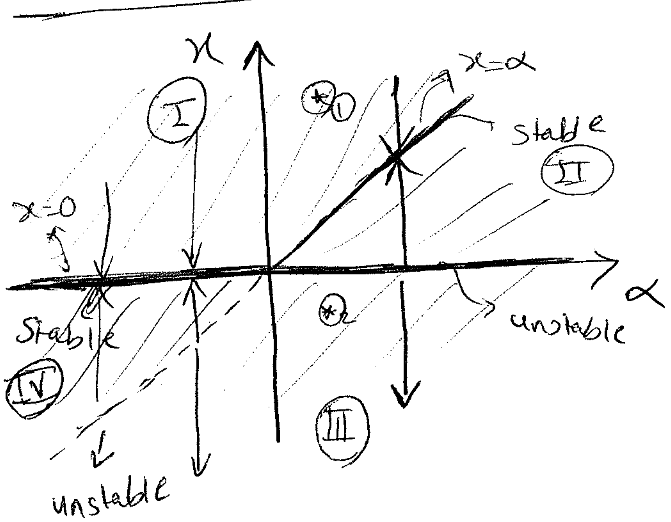
α / Eq. point	$\alpha < 0$	$\alpha > 0$
$\bar{x} = 0$	stable	unstable
$\bar{x} = \alpha$	unstable	stable

(locally assymptotically stable)



it changes stability properties!

Bifurcation Diagram:



according to your initial position wrt these regions, by having a fixed α, we can predict where we'll go!

[Ex: $\alpha > 0$, starting from $*_1$ or $*_2$ will go to different places]

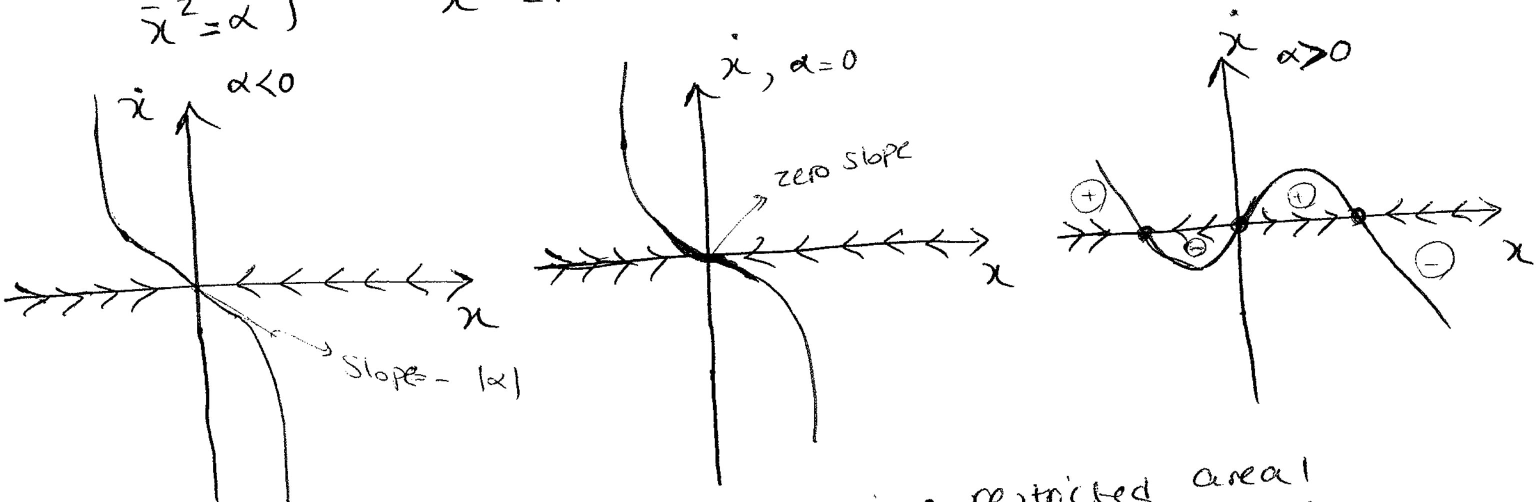
pitchfork

(a) $\dot{x} = \alpha x - x^3$, super } critical

(b) $\dot{x} = \alpha x + x^3$, sub }

(a) $f(x) = \alpha x - x^3 = x(\alpha - x^2)$

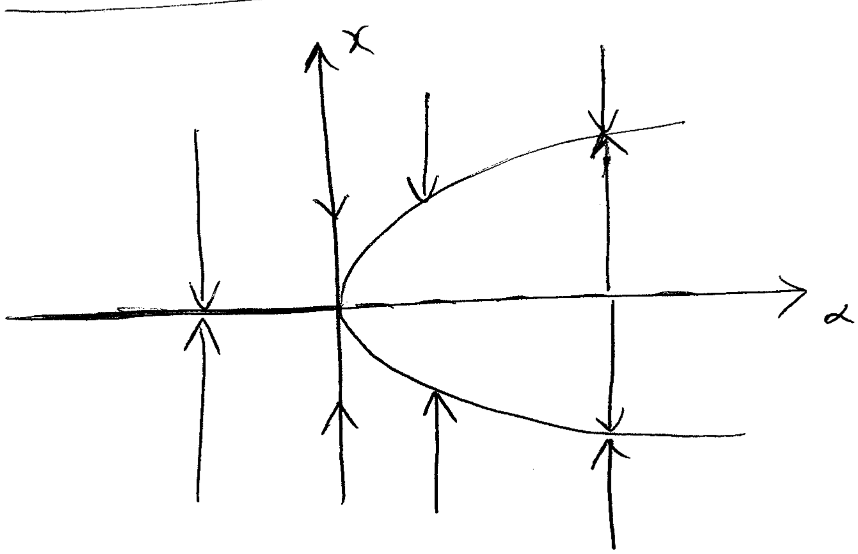
$$\left.\begin{aligned}\bar{x} &= 0 \\ \bar{x}^2 &= \alpha\end{aligned}\right\} \Rightarrow \begin{aligned}\bar{x} &= 0 \\ \bar{x} &= \pm\sqrt{\alpha}\end{aligned} \quad , \quad \alpha > 0$$



→ It's good because x is in a restricted area!

If $|\alpha|$ is not very big, x will not go so far!

Bifurcation Diagram:

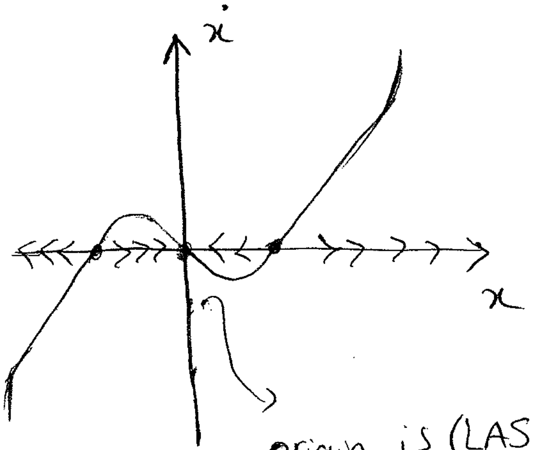


(b) $f(x) = (\alpha + x^2)\,x$

$\bar{x} = 0$

or $\bar{x} = \pm\sqrt{|\alpha|}$, $\alpha < 0$

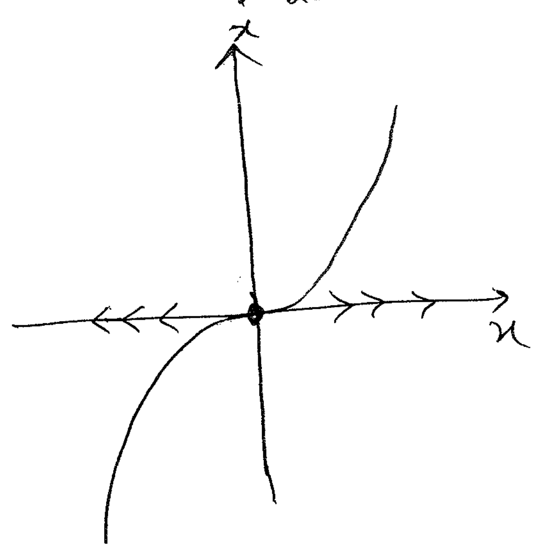
$\alpha < 0$



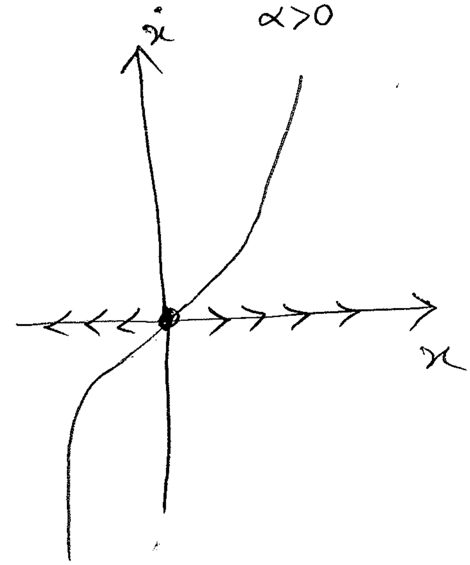
origin is (LAS)

"region of attraction is big"

$\alpha = 0$



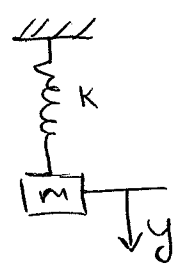
$\alpha > 0$



2nd order Systems

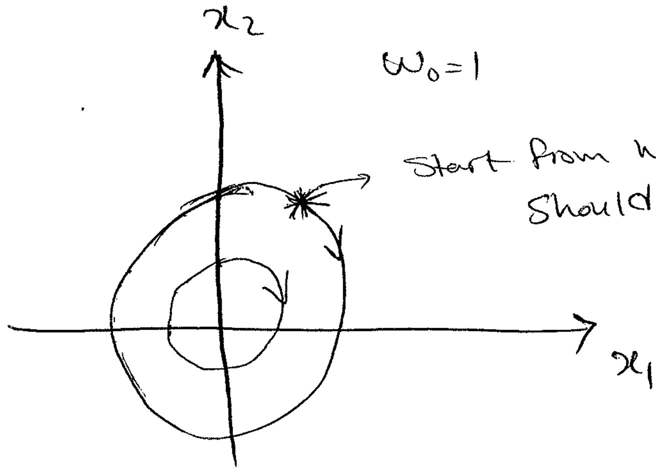
- phase portraits

Ex.



$$\ddot{y} + \underbrace{\frac{K}{m}}_{\omega_0^2} y = 0$$

$$\hookrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$\omega_0 = 1$

Start from here we have $\dot{x}_1 = x_2$ so, the circle should be clockwise!

Recall:

$\dot{x} = Ax \rightarrow$ Change of coordinate:

$$\left.\begin{array}{l} z = T \cdot x \\ \Updownarrow \\ x = T^{-1} z \end{array}\right\} \begin{array}{l} \rightarrow \dot{z} = T\dot{x} = TAx = TAT^{-1} \cdot z \\ \\ \rightarrow \dot{z} = \bar{A} \cdot z \quad \text{where } \boxed{\bar{A} = TAT^{-1}} \end{array}$$

Fact: for $n = 2$ (second order systems)

$\rightarrow$ three interesting cases:

1°) $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ $\rightarrow \bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

2°) $\lambda_{12} = \alpha \pm j\beta$ $\rightarrow \bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

3°) $\lambda_1 = \lambda_2 \in \mathbb{R}$ $\rightarrow \bar{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

+ "tech. condition"

(a) $\lambda_1 < \lambda_2 < 0$; Real

$z_i(t) = z_i(0)\, e^{\lambda_i t}$

$z_2(t) = c\, e^{\frac{\lambda_2}{\lambda_1} t}$

$c = \dfrac{z_2(0)}{(z_1(0))^{\frac{\lambda_2}{\lambda_1}}}$