Lecture 06

Feb. 13 /2014

last time:

- phase portraits of 2nd order systems
- Hartman - Grobman Thm

Today:

- Bendixon
- Poincare - Bendixon

$\}$ Thms

If time permits

- Hopf bifurcation

Super $\{$ critical ☺
sub $\}$ ☹

Bendixon Thm

applies to 2nd order systems

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

it establishes conditions for the ABSENCE of periodic orbits
(closed trajectories)

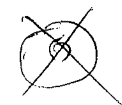
- limit cycles → (Van der pol)
- periodic trajectories (seen e.g. in harmonic oscillator)

~~If~~ Aside: If an eq. point is globally asymptotically stable then there are <u>no</u> periodic orbits!

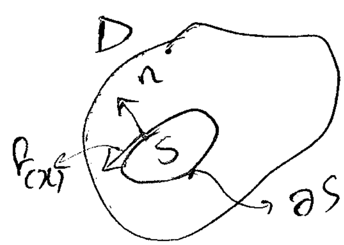
$$\operatorname{div} f = \nabla \cdot f = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

If $\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is <u>not</u> identically equal to zero or it doesn't change sign within the <u>simply connected</u> domain D of $\mathbb{R}^2$ (plane) then, 2nd order system does not have any periodic orbits.

<u>Simply connected</u>: domain without any holes.



<u>proof</u>: Assume that there is a periodic orbit in D



<u>Green's Thm</u>

$$\int_{\partial S} \underbrace{f(x) \cdot n}_{=0} \, dl = \iint_S \operatorname{div} f \, ds$$

(we assumed there is a periodic orbit)

So, if div f is not equall to zero or doesn't change sign we cannot have a periodic orbit.

Example 1 : Linear System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$\operatorname{div} f = a + d = \operatorname{trace}(A) = \text{const}$

$\Big\} \rightarrow$ only if $\operatorname{trace}(A) = 0$ Could we have periodic orbits!

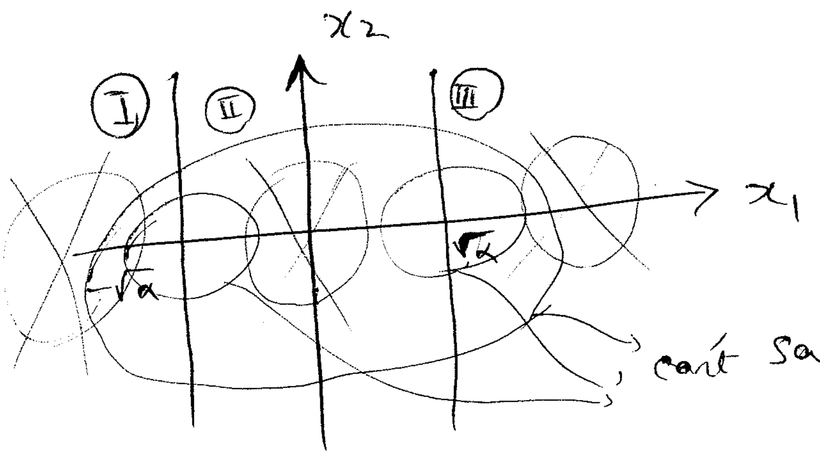
$$\bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \rightarrow \operatorname{trace}(A) = 2\alpha \rightarrow \alpha = 0 \text{ we have periodic orbit!}$$

$\rightarrow$ Bendixon : $\alpha \neq 0 \rightarrow$ no periodic orbits!

Example 2 :

$$\begin{aligned} \dot{x}_1 &= x_2 = f_1 \\ \dot{x}_2 &= -\alpha x_2 + x_1 - x_1^3 + x_1^2 x_2 = f_2 \end{aligned} \Bigg\} , \quad \underline{\alpha > 0}$$

$\operatorname{div} f = -\alpha + x_1^2$ (if α was negative or zero $\rightarrow$ no periodic orbit)



can't say! (based on Bendixon)

If Aside :

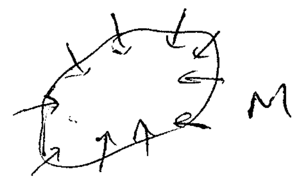
given $\dot{x} = f(x)$

$\Phi(\cdot, x_0)$ will denote trajectory . Starting @ initial condition x_0 $(\varphi(\cdot, x_0))$

Then, set M is positively (negatively) invariant if

$x_0 \in M \Rightarrow \phi(t, x_0) \in M$, for all $(t>0)$ $(t<0)$

How f behaves on the boundary of M

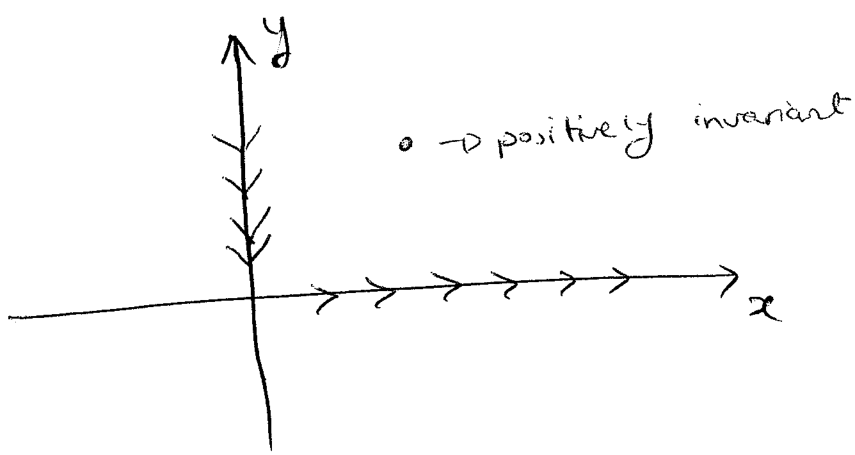


$f(x)$ should not point out of the set M (at its boundary)

predator - prey model : <u>Example</u>:

$\dot{x} = (a-by)x$: prey ; a, b, c, d positive #'s

$\dot{y} = (cx-d)y$: predator



$f_1(x, y=0) = ax$

$f_2(x=0, y) = -dy$

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vector field at the boundary of the quadrant doesn't point out of it ⇒ positively invariant.

Ex

$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) = f_1$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2) = f_2$$
$$= \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Show that :

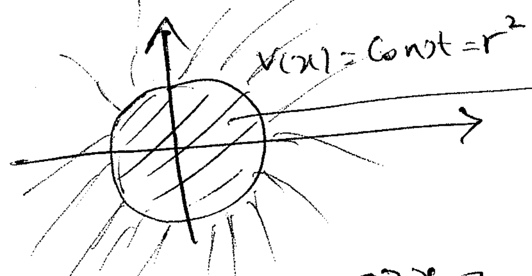
$$B_r = \{ x \in \mathbb{R}^2, \; x_1^2 + x_2^2 \le r^2 \}$$

is positively invariant for sufficiently large value r.

(to be determined)

Define:

$$V(x) = x_1^2 + x_2^2$$



$\boxed{\text{inside is the invariant set provided that } r \geq \frac{3}{2}}$ → invariant

$$\nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

Inner product

$$f^T_{(x)} \nabla V(x) = [f_1 \;\; f_2] \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} =$$
$$= 2(x_1 f_1 + x_2 f_2)$$

$$= 2\left[x_1^2 + x_1 x_2 - x_1^2(x_1^2 + x_2^2) - 2x_1 x_2 + x_2^2 - x_2^2(x_1^2 + x_2^2) \right]$$

$$= -2x_1 x_2 + 2(x_1^2 + x_2^2)(-x_1^2 - x_2^2 + 1)$$

$$= -2x_1 x_2 - 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1)$$

$(a+b)^2 \geq 0 \rightarrow \pm 2ab \le a^2 + b^2$

$$\le x_1^2 + x_2^2 - 2(x_1^2 + x_2^2)[x_1^2 + x_2^2 - 1] = -(x_1^2 + x_2^2)[2(x_1^2 + x_2^2) - 2 - 1]$$

$$= -(x_1^2 + x_2^2)\underline{[2(x_1^2 + x_2^2) - 3]} \rightarrow \text{if positive}$$

Thus, $f^T(x)\nabla V(x) \leq 0$ if $\boxed{x_1^2+x_2^2 \geq \frac{3}{2}}$

$\boxed{r \geq \frac{3}{2}}$

So, the inside of the circle is the invariant set!