Lecture 08 Feb. 20/2014

Last time:

- poincare _ Bendixon ThM
- Examples

Hopf Bifurcation

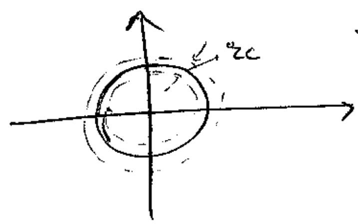
- Intro (Supercritical)

Today

- Finish Hopf (Subcritical)
- Non-dimensionalization
- Center Manifold Theory

Limit Cycle: an isolated closed trajectory V.S.

Eg. : Harmonic oscillator



there is another closed trajectory arbitrarily close to it!

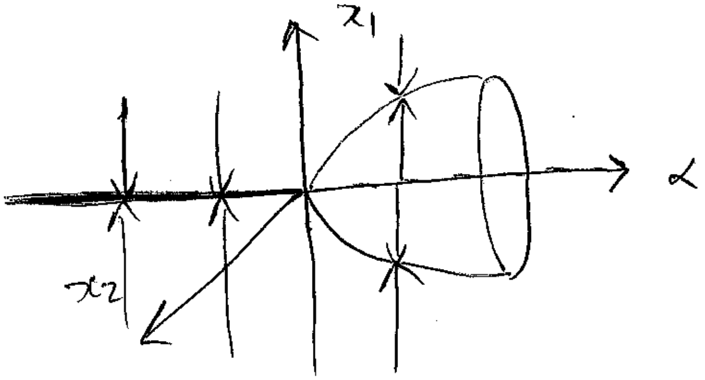
(Not a limit cycle)

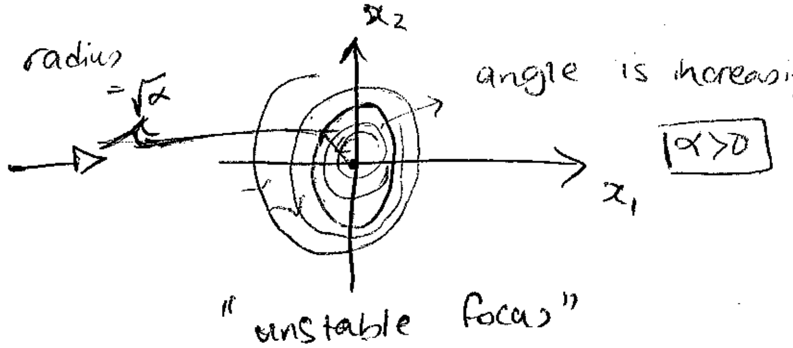
Super critical :

$$\begin{cases} \dot{r} = \alpha r - r^3 \\ \dot{\theta} = 1 \end{cases} \rightarrow \dot{r} = f(r) \rightarrow f(\bar{r}) = 0 \Leftrightarrow r = \begin{cases} 0 \\ \text{or} \\ \sqrt{\alpha}, \text{ if } \alpha > 0 \end{cases}$$

it's a limit cycle ← $\sqrt{\alpha}$

because θ is spinning around! → (this can never be an eq. point)



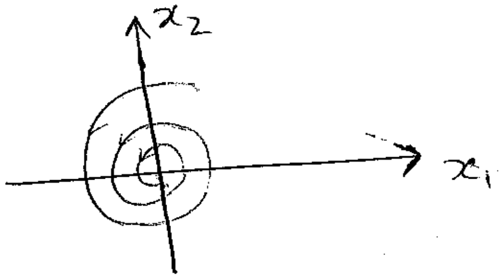


$$\dot{x}_1 = x_1(\alpha - x_1^2 - x_2^2) - x_2$$

$$\dot{x}_2 = x_2(\alpha - x_1^2 - x_2^2) + x_1$$

$\rightarrow$ linearization $\Rightarrow$ $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \boxed{\lambda_{12} = \alpha \pm j}$

Fix $\alpha < 0$



Sub critical Hopf:

$$\begin{cases} \dot{r} = \alpha r + r^3 - r^5 \\ \dot{\theta} = 1 \end{cases}$$

$\rightarrow$ eq. points: /or limit cycles (because angle is increasing, the origin is the only eq. point)

$f(\bar{r}) = 0 \Rightarrow \bar{r}(\alpha + \bar{r}^2 - \bar{r}^4) = 0 \rightarrow \begin{cases} \bar{r} = 0 \text{ (eq. point)} \\ \bar{r}^4 - \bar{r}^2 - \alpha = 0 \xrightarrow{} q = \bar{r}^2 \end{cases}$

$\bar{r}_{12} \Rightarrow q^2 - q - \alpha = 0 \rightarrow q_{12} = \frac{1}{2}\{1 \pm \sqrt{1+4\alpha}\} \geq 0$
$= \bar{r}_{12}^2$

Solutions exist iff: $1 + 4\alpha \geq 0 \rightarrow \boxed{\alpha \geq -\frac{1}{4}}$

If $\alpha = -\frac{1}{4} \rightarrow q_{12} = \frac{1}{2} \rightarrow \boxed{\bar{r} = \frac{1}{\sqrt{2}}}$

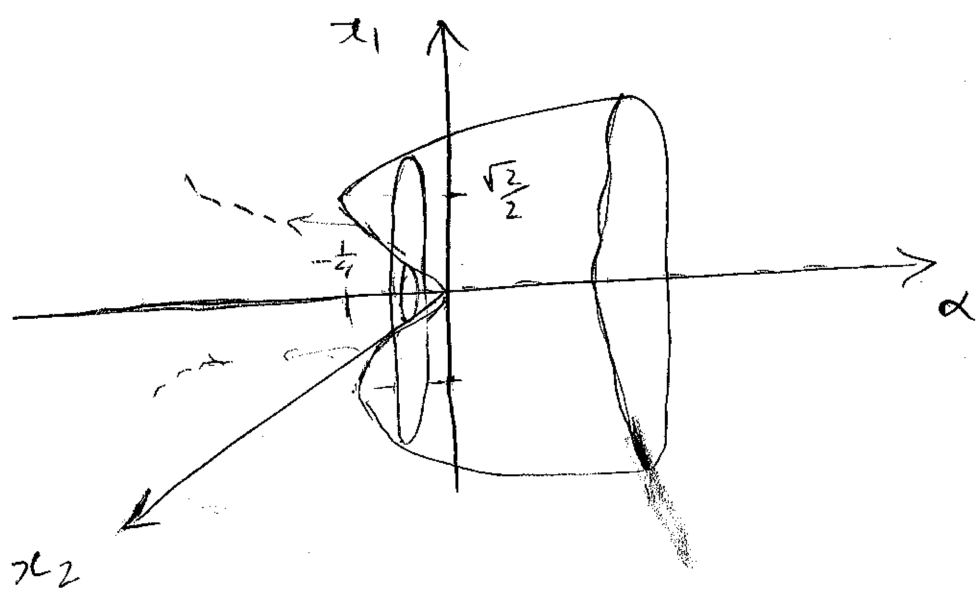
$\alpha < -\frac{1}{4} \rightarrow$ unique eq. point @ the origin!
and no limit cycles

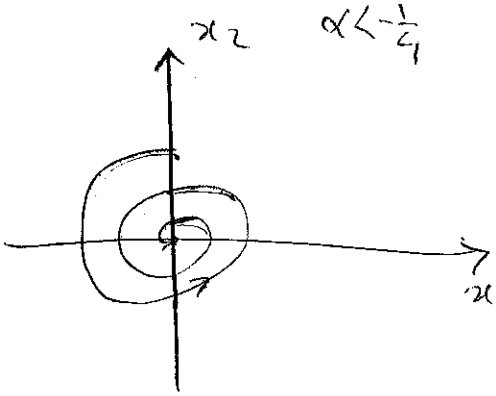
$\alpha = -\frac{1}{4} \rightarrow$ unique eq. point @ the origin!
$\rightarrow$ Limit cycle of radius $\frac{1}{\sqrt{2}}$!

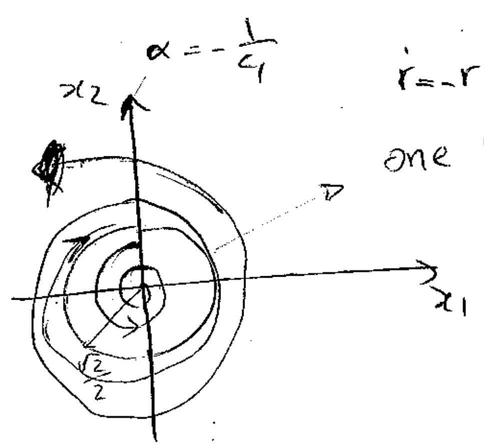
$0 > \alpha > -\frac{1}{4} \rightarrow$ unique eq. point @ the origin!
two Limit cycle $\rightarrow \bar{r}_{12} = \sqrt{\frac{1}{2}\left(1 \pm \sqrt{1+4\alpha}\right)}$.

$\alpha = 0 \rightarrow$ unique eq. point @ the origin!
Limit cycle of radius $\bar{r} = 1$!

$\alpha > 0 \rightarrow$ limit cycle of radius $\bar{r} = \sqrt{\frac{1}{2}\left(1 + \sqrt{1+4\alpha}\right)}$
unique eq. point @ the origin!

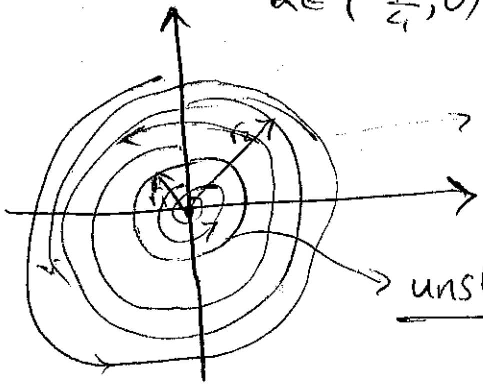






$\dot{r} = -r\left(r^2 - \frac{1}{2}\right)^2$
one limit cycle

$\alpha \in (-\frac{1}{4}, 0)$



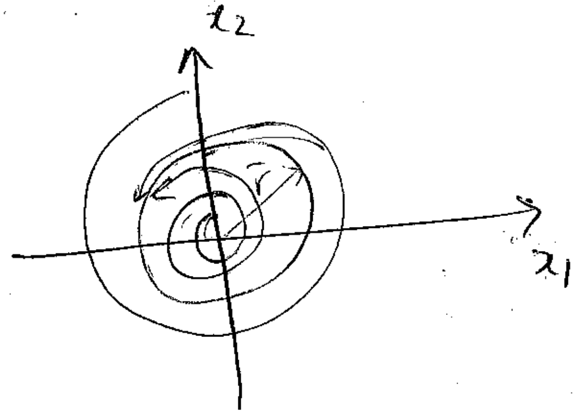
$\rightarrow$ (two limit cycles)

$\rightarrow$ unstable: r_1

(zero and $\bar{r}_2$ are stable and $\bar{r}_1$ is unstable)

$\begin{cases} \alpha \to 0 \Rightarrow \text{Smaller limit cycles shrinks to origin} \\ \text{and The bigger one's radius will be } \frac{1}{\ } ! \end{cases}$

$\alpha > 0$



Scaling :

EX:

$$\dot{x}_1 = -\alpha x_1 + \beta x_2$$

$$\dot{x}_2 = \frac{\gamma x_1}{\delta + x_1^2} - \eta x_2$$

Greek letters: parameters

Scaled variables :

$$z_i = \frac{x_i}{X_i}, \quad i = 1, 2$$

$$\tau = \frac{t}{T}$$

T, X_1, X_2 (constants, "scales") to be determined

(2 dependent variables and one independent variable $\rightarrow t$)

$$\frac{dx_i}{dt} = \frac{\partial \tilde{\tau}}{\partial t}\frac{\partial x_i}{\partial \tilde{\tau}} = \frac{1}{T} \times \frac{\partial x_i}{\partial \tilde{\tau}} = \frac{1}{T} X_i \frac{\partial z_i}{\partial \tilde{\tau}}$$

$$\rightarrow \frac{dx_1}{dt} = \frac{X_1}{T}\frac{dz_1}{d\tilde{\tau}} \rightarrow \quad \frac{dz_1}{d\tilde{\tau}} = \frac{T}{X_1}\left(-\alpha X_1 z_1 + \beta X_2 z_2\right)$$

$$\frac{dz_2}{d\tilde{\tau}} = \frac{T}{X_2}\left(\frac{\gamma X_1 z_1}{\delta + X_1^2 z_1^2} - \eta X_2 z_2\right)$$

Can show that:

$$\begin{cases} \dfrac{dz_1}{d\tilde{\tau}} = -a z_1 + z_2 \\[2ex] \dfrac{dz_2}{d\tilde{\tau}} = \dfrac{z_1}{1+z_1^2} - b z_2 \end{cases}$$

" with proper choice of X_1, X_2, T ! "

$\rightarrow$ only two parameters (a, b) matter!