Lecture 19

April 10 / 2014

last time:

- Back stepping (*)

Today:

- Additional Comments about (*)
- Examples
- Control Lyapunov Functions (CLF)
- Comparison functions

Ex:

$$\left.\begin{aligned} \dot{X}_1 &= X_1^2 + X_2 \\ \dot{X}_2 &= u \end{aligned}\right\}$$

objective

$x_1(t) \xrightarrow[t\to\infty]{} r(t)$ ↳ reference Signal

Step1

$z_1(t) = x_1(t) - r(t)$ → $\underline{(\to 0)}$ [change of variables]

$$\dot{z}_1(t) = \dot{x}_1(t) - \dot{r}(t) = x_1^2 + x_2 - \dot{r}(t) = (z_1(t) + r(t))^2 + x_2 - \dot{r}(t)$$

our objective is to : $z_1(t) \xrightarrow[t\to\infty]{} 0$

$$V_1(z_1) = \frac{1}{2} z_1^2$$

↳ $\dot{V}_1 = z_1 \cdot \dot{z}_1 = z_1 \left((z_1 + r)^2 - \dot{r} + x_2 \right)$

↳ x_2: virtual Control

$$x_2 = \dot{r} - \underbrace{(z_1 + r)^2}_{x_1^2} - \underbrace{K_1}_{\text{positive}} z_1 = \alpha_1(z_1, r, \dot{r})$$

with this choice of x_2:

$$\boxed{\dot{V}_1 = -K_1 z_1^2}$$

But x_2 is not control,

Fix : apply back stepping

Step 2 : $z_2 := x_2 - \alpha_1(z_1, r, \dot{r})$ → proceed as last time!

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 \quad \ldots$$

$(\dot{V}_1 = -2K_1 V_1)$

$\rightarrow V_1(t) = e^{-2K_1 t} V_1(0)$

Speed of Convergence (exponentially fast)

$$\dot{x} = f(x) + g(x)\xi$$

$$\dot{\xi} = f_a(x, \xi) + g_a(x, \xi) u$$

$$g_a(x, \xi) \neq 0 \quad \text{for all } (x, \xi)$$

Do the same thing here (cancling some terms is possible when you trust your model)

$$u = \frac{1}{g_a(x, \xi)} \{ \ldots \}$$

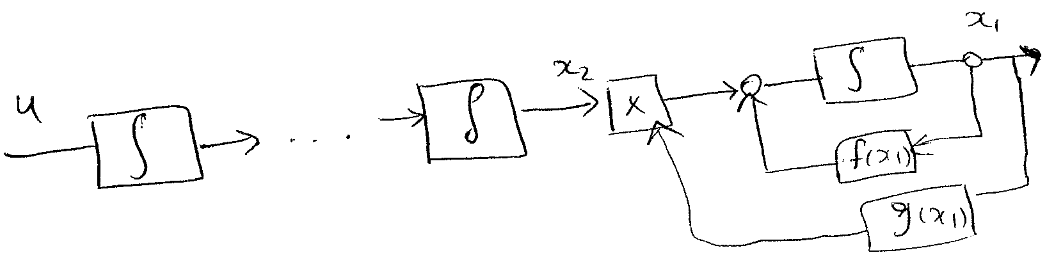
Ex :

$$\begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{cases} \rightarrow$$

x_2 as virtual control for x_1
x_3 " " for x_2
u for whole system!

$$\rightarrow \begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ \dot{x}_3 = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u \end{cases}$$

$\rightarrow$ This triangular form



Control Lyapunov Functions (CLF)

Given

$$\dot{x} = f(x) + g(x)u$$

and $V(x)$: positive definite and radially unbounded

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}\left[f(x) + g(x)u\right] = \underbrace{\frac{\partial V}{\partial x}f(x)}_{L_f V(x)} + \underbrace{\frac{\partial V}{\partial x}g(x)}_{L_g V(x)}u$$

$$\dot{V} = L_f V(x) + L_g V(x)\,u$$

$V(x)$ is CLF if $L_g V(x) = 0 \Rightarrow L_f V(x) < 0$ for all $x \neq 0$

(without $u \rightarrow L_f V(x)$ is well-behaved)

If $V(x)$ is CLF, how can we use it to construct our Controller?

There are many ways of obtaining u from CLF?

E.g. $\boxed{\text{Sontag's Formula}}$

$$u = \alpha(x) = \begin{cases} 0 & , L_g V(x) = 0 \\ -\dfrac{L_f V(x) + \sqrt{(L_f V(x))^2 + (L_g V(x))^2}}{L_g V(x)} & , \text{else} \end{cases}$$

$\alpha(x)$: Continuous function of x if "Small Control property" Satisfied. (technical Condition)

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$\dot{z} = g(z)$: time invariant

objective: Study stability of trajectory $\bar{z}(t)$!

$$\Downarrow$$

$$\dot{\bar{z}}(t) = g(\bar{z}(t))$$

$$\bar{z}(t_0) = \bar{z}(0) : \text{given}$$

Change of coordinates:

deviations → $x(t) = z(t) - \bar{z}(t)$ fixed ↳ trajectory

$z(t)$ ↑ real state

$$\dot{x}(t) = \dot{z}(t) - \dot{\bar{z}}(t) = g(z(t)) - g(\bar{z}(t)) \Rightarrow$$

$$\dot{x}(t) = g(x(t) + \bar{z}(t)) - g(\bar{z}(t)) =: f(x, t)$$

$\bar{z}(t)$ results in t explicitly exists! $\rightarrow$ $\bar{x} = 0$ is eq. point!