Lecture 21

April 22 / 2014

Last time

- Comparison functions

$K: \alpha: \mathbb{R}_+ \to \mathbb{R}_+$

$[0,a) \to [0,b)$

K, K_∞, K_L

$\left.\begin{array}{l}\alpha(0)=0 \\ \alpha \uparrow\end{array}\right\}$ $\oplus$ $\alpha(r) \xrightarrow{r\to\infty} \infty$

$$\begin{cases} B(r,s) \\ B(\cdot,s): K \quad (\text{class } K) \\ B(r,\cdot): \text{decreasing} \\ \quad \lim_{s\to\infty} B(r,s) = 0, \ \forall r \end{cases}$$

- Stability of time-varying systems

uniform stability: $\|x(t)\| \leq \alpha \|x_0\|$

uniform asymptotic stability: $\|x(t)\| \leq B(\|x_0\|, t-t_0)$

In particular, if $B(r,s) = kre^{-as}$, $k, a > 0$ $\to$ K_L function

$\Downarrow$

uniform exponential stability

Ex: In linear case:

$\dot{x} = A(t)\,x$

$x(t) = \varphi(t,t_0)\,x(t_0)$

"State transition matrix"

$\Rightarrow \|x(t)\| = \|\varphi(t,t_0) \times x(t_0)\|$

$\leq \|\varphi(t,t_0)\|\,\|x(t_0)\|$

$$\begin{cases} \dfrac{\partial \varphi(t,t_0)}{\partial t} = A(t)\,\varphi(t,t_0) \\ \varphi(t_0,t_0) = I \end{cases}$$

In order to have ~~uniform~~ ~~asymptotic~~ exponential stability

$$\Rightarrow \boxed{\|\varphi(t,t_0)\| \leq k\, e^{-a(t-t_0)}}$$

$$\Rightarrow \|\varphi(t,t_0)\|\|x(t_0)\| \leq k\, e^{-a(t-t_0)} \|x(t_0)\|$$

Question: $\dot{x} = A(t)\,x$

$$\underset{\forall t}{\underset{\forall i}{Re(\lambda_i(A(t))}} \overset{?}{\not\Rightarrow} \text{Exp. Stability} \qquad \boxed{\text{No!}}$$

Ex: $A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ 1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}$

$$\lambda_{1,2}(A(t)) = -\frac{1}{4} \pm j\frac{\sqrt{7}}{4}$$

$$\varphi(t,0) = \left[\begin{array}{c|c} e^{t/2}\cos t & * \\ \hline * & * \end{array}\right] \qquad e^{t/2} \rightarrow \|\varphi(t,0)\| \rightarrow \infty$$

Although all eigenvalues are on LHP, we don't have exp. stability!

Big Theorem :

$$\begin{cases} \dot{x} = f(x,t) \\ f(0,t) = 0 \end{cases}$$

① If

$$w_1(x) \leq V(x,t) \leq w_2(x) \quad (1)$$

and $$\dot{V}(x,t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq 0 \quad (2)$$

for some positive definite functions "w_1, w_2" on domain D that includes the origin ($\bar{x}=0$), then $\bar{x}=0$ is uniformly stable.

② If further $\dot{V}(x,t) \leq -w_3(x)$, $\forall x \in D$ for some positive definite w_3, then $\bar{x}=0$ is uniformly asymptotically stable.

③ If $D = \mathbb{R}^n$ and w_1 is radially unbounded, then $\bar{x}=0$ is globally uniformly asymptotically stable.

④ If $w_i(x) = K_i \|x\|^a$ for some $K_1, K_2, K_3 > 0$ and $a > 0$ then $\bar{x}=0$ is exponentially stable.

Aside Ex:

$$\dot{x} = -x^3$$

$$V(x) = \frac{1}{2} x^{2}$$

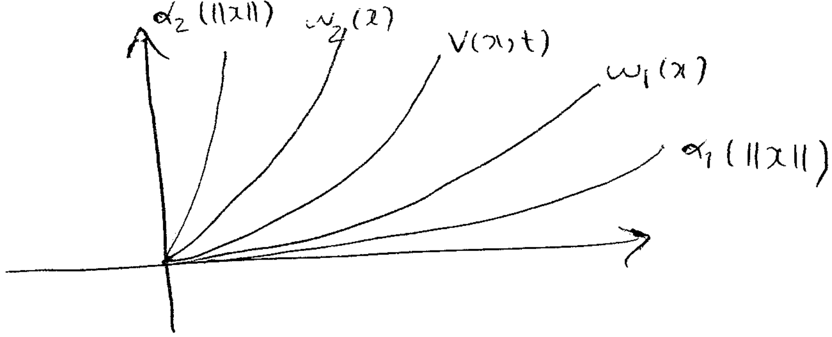
$$\dot{V} = x\dot{x} = -x^{4}$$

→ is not exponentially stable

(more general examples soon)

moreover, linearization stable → (A Hurwitz) (in time inv. case) $\overset{\text{not}}{\Rightarrow}$ Exp. Stability (of $\bar{x}=0$)

key



Fact: Any positive definite function can be bounded by

$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$

α_i : class K functions

If Ex: $V(x) = x^T P x \rightarrow \alpha_{i(r)} = \lambda_k(P) r^2$; $k = \begin{cases} \min, & i=1 \\ \max, & i=2 \end{cases}$

proof:

1 Eq. (1) $\Rightarrow \exists\, \alpha_1, \alpha_2$: Class K

s.t. $\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|)$ for all t

Eq. (2) $\Rightarrow \dfrac{dV(x,t)}{dt} \le 0 \Rightarrow V(x,t) \le V(x_0, t_0)$

(non-increasing function)

$\alpha_1(\|x\|) \le V(x,t) \le V(x_0,t_0) \le \alpha_2(\|x_0\|)$

$\Rightarrow \alpha_1(\|x\|) \le \alpha_2(\|x_0\|)$

we want to show $\left[\|x\| \le \alpha(\|x_0\|)\right] \rightarrow$ [for uniformly stable]

$$\boxed{\|x\| \le \alpha_1^{-1}(\alpha_2(\|x_0\|))}$$

Remaining Task: Show that $\alpha_1^{-1} \circ \alpha_2$ is class K !

Facts: (lemma 4.2 Khalil) $\quad$ (Ex: $\alpha(r) = r^2 \rightarrow \alpha^{-1}(r) = \sqrt{r}$)

① If α is $K_{[0,a]}$ $\rightarrow$ $\alpha^{-1} \in K_{[0,a)}$

② $\alpha_1 \circ \alpha_2$ where $\alpha_1, \alpha_2 \in K$ $\rightarrow$ $\alpha_1 \circ \alpha_2 \in K$!

② $\quad V(x,t) \leq \alpha_2(\|x\|)$

$\alpha_3 \quad (\Rightarrow \alpha_2^{-1}(V(x,t)) \leq \|x\|)$

$\dot{V}(x,t) \leq -\omega_3(x) \quad \leq -\alpha_3(\|x\|)$

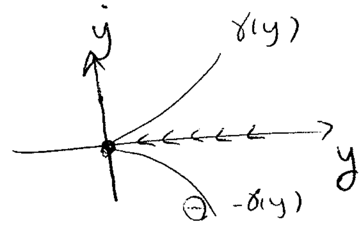
(because ω_3 is PD so we have lower bound for it $\alpha_3 \leq \omega_3 \leq \alpha_4$)

$\Rightarrow \alpha_3(\alpha_2^{-1}(V(x,t))) \leq \alpha_3(\|x\|) \Rightarrow -\alpha_3(\|x\|) \geq -\alpha_3(\alpha_2^{-1}(V(x,t)))$

$\Rightarrow \quad \dot{V}(x,t) \leq -\omega_3(x) \leq -\alpha_3(\|x\|) \leq -\underbrace{\alpha_3(\alpha_2^{-1}(V(x,t)))}$

class K function

$\Rightarrow \dot{V}(x,t) \leq -\alpha(V(x,t))$



($\gamma(0) = 0$, increasing)

$\dot{y} = -\gamma(y)$

$y \in \mathbb{R}_+$, γ is class K function

Fact: $y(t) = \beta(y_0, t - t_0)$ (y can be written like this)

β: class K_L function

$\Rightarrow \boxed{V(x,t) \leq \beta(V(x_0,t_0), t - t_0)}$

Ex:
$$\dot{x} = -g(t)\,x^3$$
$$g(t) \geq 1 \quad \text{for all } t$$

$$V(x) = \frac{1}{2}x^2 \implies \dot{V}(x) = x\dot{x} = -g(t)\,x^4 \leq -x^4$$

$\| = W_1, = W_2$ $\quad$ $\boxed{, W_3 = x^4}$

difference in powers: Not Exp. Stable!

$\rightarrow \boxed{\text{GUAS}}$

time is completely eliminated: uniformly