Due Tu 03/22/16 (at the beginning of the class)

1. Khalil, Problem 4.14 (attached).

2. What kind of equilibrium stability (stable (in the sense of Lyapunov), or AS, or GAS) if any, is exhibited by the state representation of

   (a) The $\frac{1}{s^2}$ plant with no input, i.e. $\dot{x}_1 = x_2$, $\dot{x}_2 = 0$.

   (b) The magnetically suspended ball: $\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& \frac{-c}{m}\frac{\bar{u}^2}{x_1^2} + g \end{array}$ with $\bar{u} = \sqrt{\frac{mg}{c}Y} = \text{const.}$

3. The Morse oscillator is a model that is frequently used in chemistry to study reaction dynamics. The equations for an unforced Morse oscillator are given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\mu(e^{-x_1} - e^{-2x_1}). \end{aligned}$$

   (a) Find the equilibrium points of the system.

   (b) Investigate their stability properties.

4. Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= -\frac{x_2}{1+x_1^2} - 2x_1, \\ \dot{x}_2 &= \frac{x_1}{1+x_1^2}. \end{aligned}$$

   (a) Show that the origin is an equilibrium point.

   (b) Using the candidate Lyapunov function

$$V(x) = x_1^2 + x_2^2,$$

   what are the stability properties of the equilibrium point?

   (c) Linearize the nonlinear system around the equilibrium point.

   (d) What can you deduce about the stability properties of the origin based on linearization?

   (e) Obtain a suitable Lyapunov function by solving the Lyapunov equation

$$A^T P + PA = -Q,$$

   where

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

5. Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\,g(k_1 x_1 + k_2 x_2), \qquad k_1,\, k_2 > 0, \end{aligned}$$

   where the nonlinearity $g(\cdot)$ is such that

$$g(y)\,y > 0, \quad \forall\, y \neq 0$$

$$\lim_{|y|\to\infty} \int_0^y g(\xi)\,\mathrm{d}\xi = +\infty$$

   (a) Using an appropriate Lyapunov function, show that the equilibrium $x = 0$ is globally asymptotically stable.

   (b) Show that the saturation function $\mathrm{sat}(y) = \mathrm{sign}(y)\min\{1, |y|\}$ satisfies the above assumptions for $g(\cdot)$. What is the exact form of your Lyapunov function for this saturation nonlinearity?

(c) Parts (a) and (b) imply that a double integrator with a saturating actuator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \text{sat}(u) \end{aligned}$$

can be stabilized with the state-feedback controller $u = -k_1 x_1 - k_2 x_2$. Design k_1 and k_2 to place the eigenvalues of the linearization at $-1 \pm j$, and simulate the resulting closed-loop system both with, and without, saturation. Compare the resulting trajectories. (Please provide plots of $x_1(t)$ and $x_2(t)$ rather than phase portraits.)

(a) Show that $V(x) \to \infty$ as $\|x\| \to \infty$ along the lines $x_1 = 0$ or $x_2 = 0$.

(b) Show that $V(x)$ is not radially unbounded.

4.10 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x\, d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x)Pf(x)$ is positive definite for all $x \in R^n$ and radially unbounded.

(c) Show that the origin is globally asymptotically stable.

4.11 Using Theorem 4.3, prove Lyapunov's first instability theorem: For the system (4.1), if a continuously differentiable function $V_1(x)$ can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along the trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.12 Using Theorem 4.3, prove Lyapunov's second instability theorem: For the system (4.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\lambda > 0$ and $W(x) \geq 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

4.13 For each of the following systems, show that the origin is unstable:

$$(1) \quad \dot{x}_1 = x_1^3 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

$$(2) \quad \dot{x}_1 = -x_1^3 + x_2, \qquad \dot{x}_2 = x_1^6 - x_2^3$$

Hint: In part (2), show that $\Gamma = \{0 \leq x_1 \leq 1\} \cap \{x_2 \geq x_1^3\} \cap \{x_2 \leq x_1^2\}$ is a nonempty positively invariant set, and investigate the behavior of the trajectories inside Γ.

4.14 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(x_1)(x_1 + x_2)$$

where g is locally Lipschitz and $g(y) \geq 1$ for all $y \in R$. Verify that $V(x) = \int_0^{x_1} y g(y)\, dy + x_1 x_2 + x_2^2$ is positive definite for all $x \in R^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.