Lecture 1:

- Nonlinear Dynamics and Chaos w/ Applications to Physics, Biology, Chemistry & Engineering.
  (Steven Strogatz)

- Nonlinear Dynamical Systems
  - Static eq^n : algebraic eq^n
    eg: $f(x) = 0$
  - Dynamical eq^n : $\frac{dx}{dt} = f(x)$ ------- (1)
    LHS: rate of change of quantity x RHS: nonlinear function of x

    t: time
    $\frac{d}{dt}$: derivative wrt time
    $x(t)$: state vector :
    f: nonlinear function of x

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad x_i(t) \in \mathbb{R} \qquad x(t) \in \mathbb{R}^n \quad (\mathbb{R}^{n\times 1})$$

In this course:

↳ Study analysis of systems of the form $\frac{dx}{dt} = f(x)$ (1) or its time varying version:
$\frac{dx}{dt} = f(x,t)$ ——— (2)

↳ Also systems w/ inputs: $\frac{dx}{dt} = f(x,u)$ (3) where $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathbb{R}^m$

input could be disturbance (exogenous) or control (to be designed)

If (3) can be converted to (1); easier

Note: If $u = u(x)$ (3) ⟶ (1)

Many tools for analysis will be useful in the context of synthesis (i.e. control design)

- In EE 5231 | Linear Systems ⟶
  (1) simplifies to: $\frac{dx}{dt} = Ax$ ($\because f(x) = Ax$ is a linear function of x)
  not $Ax + b$: affine
  (3) simplifies to: $\frac{dx}{dt} = Ax + Bu$ $A \in \mathbb{R}^{n\times n}$ → generates dynamics
  $B \in \mathbb{R}^{n\times m}$ → introduces exogeneous inputs
  both state & input enter linearly

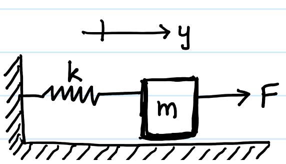
- Linear ⟶ Nonlinear
  loses superposition.

- i.e. for linear systems, superposition holds

$$x(t) = \underbrace{e^{At}x(0)}_{\text{contribution of IC}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{contribution of input}}$$

also holds for time-varying cases where A and B are time-varying matrices (state transition matrices).

Eg: (1):



$$m\frac{d^2y}{dt^2} + ky = F \quad -(*)\Big\}$$ Input-output differential equation

F : external force (input) u $\longrightarrow$ no derivatives of input w.r.t. time in ⊛

y : position (output) $\Rightarrow \therefore$ can choose : $x_1(t) = y(t)$ $\qquad \dot{x}_1 = \dot{y} = x_2$

$x_2(t) = \dot{y}(t)$ $\qquad \dot{x}_2 = \ddot{y} = -\frac{k}{m}x_1 + \frac{1}{m}u$

From ⊛ : $\ddot{y} = -\frac{k}{m}x_1 + \frac{1}{m}u$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 + \frac{1}{m}u \end{bmatrix} = f(x,u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix}$$

In this case, clearly, we have a linear system

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u$$

$$y = x_1 \Rightarrow y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

State space representation

- Choose physical states from 0 to $(n-1)^{th}$ derivative. if no derivatives of input present
  $y, \dot{y}$ for 2^{nd} order ODE

➤ Eg: 2 $\quad \dot{x} = \sin(x) \qquad x(t) \in \mathbb{R}$; scalar

- Look at equilibrium points, $\bar{x}$: solutions of $f(\bar{x}) = 0$ i.e. $\frac{d\bar{x}}{dt} = 0$
  i.e. "you start at $\bar{x}$ $\Rightarrow$ you stay there forever (i.e. $\forall$ time $t \geq 0$)"

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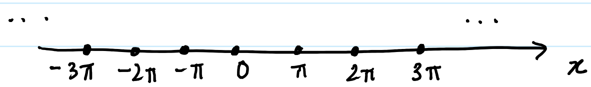
↳ In the linear case : $A\bar{x} = 0$

$\Rightarrow \bar{x} = 0$ is always an equilibrium point

- If A is invertible $\Rightarrow \det(A) \neq 0 \Rightarrow \bar{x} = 0$ is a unique equ^m pt.
- If A is non-invertible $\Rightarrow \det(A) = 0 \Rightarrow \bar{x} \in \mathcal{N}ull(A) = \{z \in \mathbb{R}^n ; Az = 0\}$
  can compute null space by singular value decomposition SVD.

$\therefore$ Either unique equ^m point : $\bar{x} = 0$ or infinitely many of them $(\mathcal{N}ull(A))$
(entire subspace of equ^m pt.s)

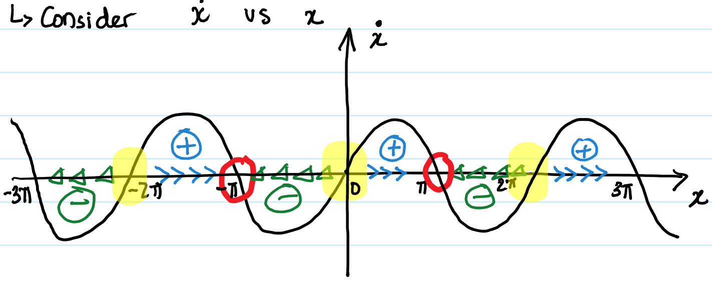
Equ^m pt : $\sin(\bar{x}) = 0 \Rightarrow \bar{x} = k\pi \quad k = 0, \pm 1, \pm 2, \dots$



infinitely many, isolated equ^m pts

➤ stable or unstable equ^m?

↳ Consider $\dot{x}$ vs x



O stable. equ^{m}s

unstable equ^{m}s.

$$\frac{df(x)}{dx} = \cos(x)\Big|_{x=\bar{x}} = \begin{cases} \cos(2n\pi) = +1 \longrightarrow k \text{ even} \\ \cos(n\pi) = -1 \longrightarrow k \text{ odd} \end{cases}$$