EE 8215 S'16: Nonlinear Systems
Lecture 4: Jan 28th 2016, Thu

▶ Last time: 1) Fold bifurcation: $\dot{x} = \alpha \pm x^2$

2) Transcritical bifurcation: $\dot{x} = \alpha x \mp x^2$

▶ Today:

3) Pitchfork bifurcation

$\dot{x} = \alpha x \mp x^3$ $\qquad$ $\dot{x} = \alpha x - x^3$ ⟹ Supercritical

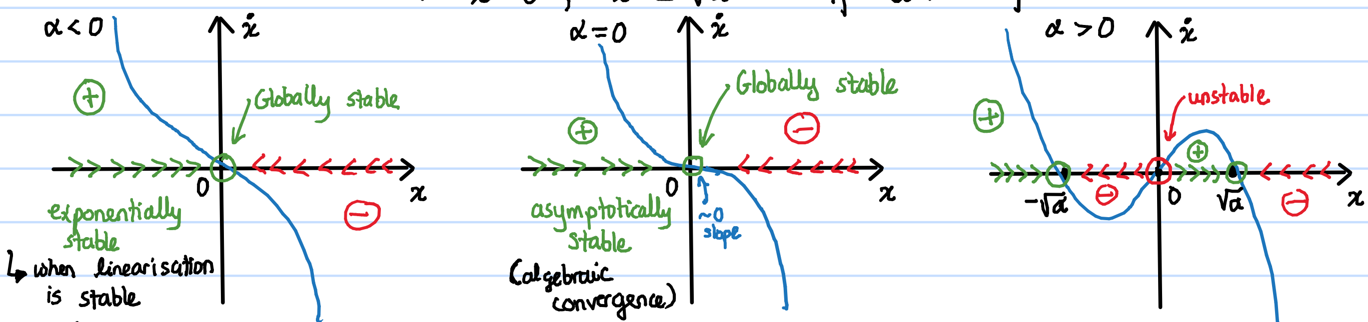
$\dot{x} = \alpha x + x^3$ ⟹ Subcritical

↳ Supercritical

Equ^m points: $\dot{x} = 0 \Rightarrow \alpha\bar{x} - \bar{x}^3 = 0$ $\qquad$ $\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \alpha - 3\bar{x}^2$

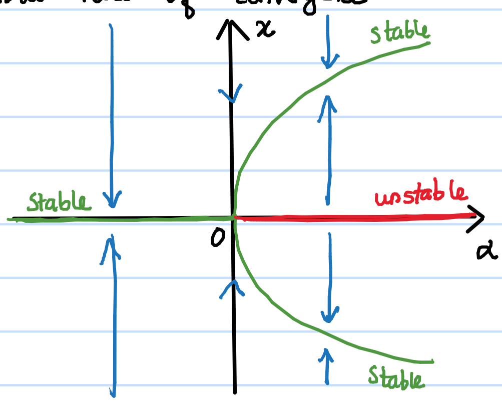
$\bar{x}(\alpha - \bar{x}^2) = 0$

→ $\bar{x} = 0$, $\bar{x} = \pm\sqrt{\alpha}$ if $\alpha > 0$



↳ "exp. stable" → exponential rate of convergence $\qquad$ Eg: air conditioning

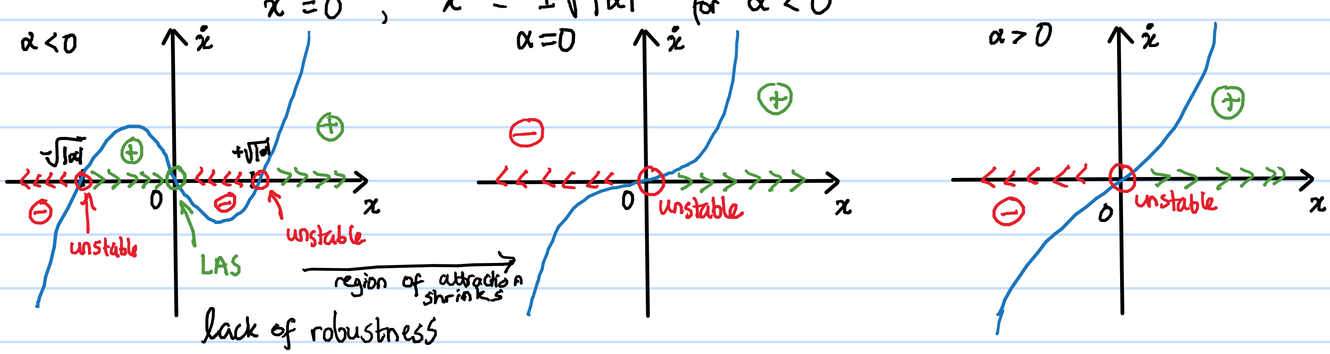
Bifurcation diagram:



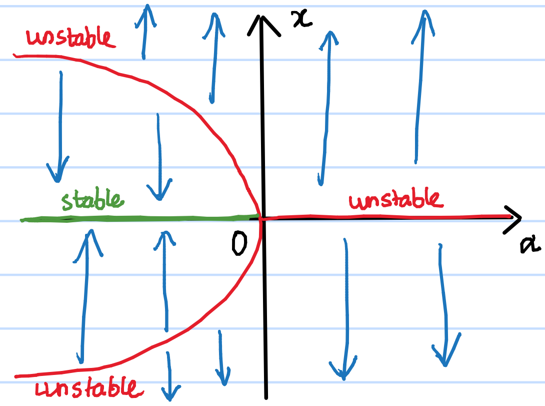
↳ Subcritical

Equ^m pts: $\bar{x}(\alpha + \bar{x}^2) = 0$

$\bar{x} = 0$, $\bar{x} = \pm\sqrt{|\alpha|}$ for $\alpha < 0$



Bifurcation diagram:

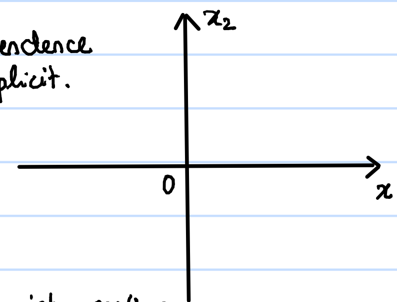


➢ Phase Portraits

- (Visualisation of 2^{nd} order systems)

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2$$

Time-dependence implicit.



• Consider linear systems:

$\dot{x} = Ax$ $\quad A_{2\times2}$

eigenvalues / eigenvectors → split x_1-x_2 plane into sections

↳ whether x_1, x_2 approach zero

- Coordinate transformation: $x = Tz$ $\quad T$ is invertible

(can go back & forth) $\dot{z} = T^{-1}x = \underbrace{T^{-1}AT}_{\bar{A}}z$ $\Big\}$ $\dot{z} = \bar{A}z$

Three special cases:

(a) $\bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\quad \lambda_1, \lambda_2 \in \mathbb{R}, \; \lambda_1 \neq \lambda_2$ (diagonalise 2x2 matrix)

+ equations uncoupled

(b) $\bar{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\quad \lambda \in \mathbb{R}$ (Jordan canonical form)

(c) $\bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ $\quad \alpha, \beta \in \mathbb{R}$ (Complex-conjugate eigenvalues)

(a) $\dot{z}_i = \lambda_i z_i \Rightarrow$ Solution: $z_i(t) = z_i(0)e^{\lambda_i t}$ $\quad i = 1, 2$

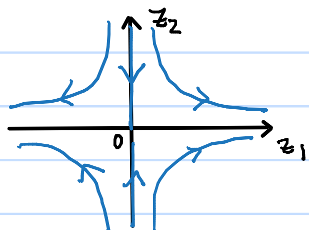
$e^{\lambda_1 t} = \dfrac{z_1(t)}{z_1(0)}$ $\qquad z_2(t) = z_2(0)\left(e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1}$

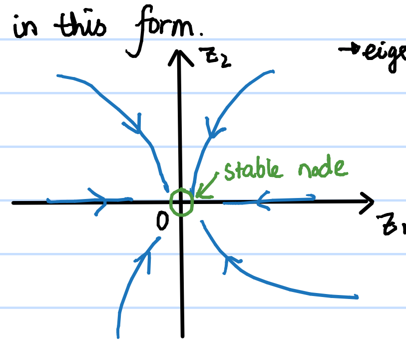
$\Rightarrow$ $z_2 = c\, z_1^{\lambda_2/\lambda_1}$ can express z_2 in this form.

where $c = \dfrac{z_2(0)}{\left(z_1(0)\right)^{\lambda_2/\lambda_1}}$

$\lambda_2 < 0 < \lambda_1$

"saddle"





→ eigenvectors are along axes

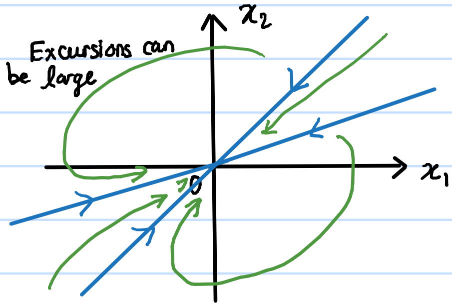
$\lambda_1, \lambda_2 < 0 \Rightarrow$ Stable convergence

If $\lambda_1, \lambda_2 >$

→ shoots to ∞

(unstable node)

divergence to $+\infty$



- rotated eigenvectors
- eigenvectors do not need to be orthogonal

__Note__ : $A^T A = A A^T \Rightarrow A$: normal

then eigenvectors of A are orthogonal

(c) Complex-conjugate eigenvalues

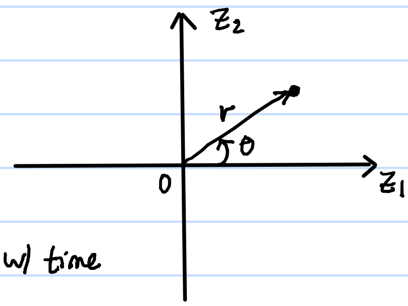
$\lambda_{1,2} = \alpha \pm j\beta$ (oscillations)

↳ Polar coordinates:

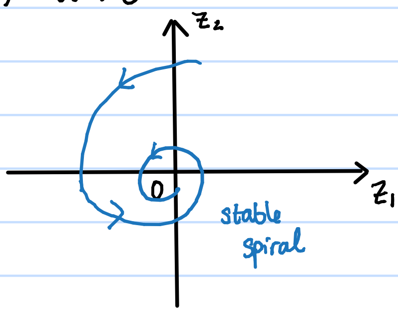
$z_1 = r\cos\theta$, $z_2 = r\sin\theta$

$\dot{r} = \alpha r$ → oscillation growing/decaying w/ time

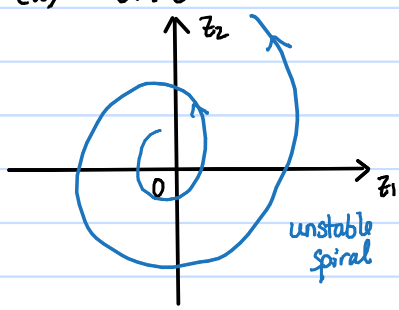
$\dot{\theta} = \beta \Rightarrow \theta(t) = \theta_0 + \beta t$



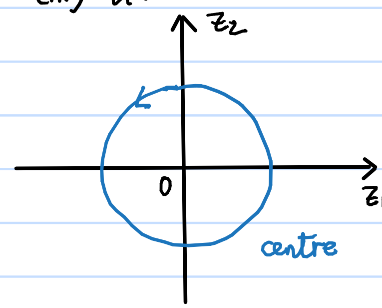
(c) (i) $\alpha < 0$



(ii) $\alpha > 0$



(iii) $\alpha = 0$



(b) Eg: $\lambda = 0 \Rightarrow \dot{x} = 0$

$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

eg: a hockey puck moving on a frictionless surface

Equ^m pts : $\bar{x}_2 = 0$, $\bar{x}_1 \in \mathbb{R}$ | infinitely values

Solutions : $\dot{x}(t) = c_1 = \dot{x}(0)$

$x(t) = c_1 t + c_2$

$\hookrightarrow x(0)$