EE 8215 : Nonlinear Systems
Lecture 7 : Feb 11, 2016, Thu

➤ **Last time**: - Poincaré - Bendixson Thm
- Supercritical Hopf bifurcation :)

Eg: $\dot{r} = \alpha r - r^3$

$\dot{\theta} = 1$

➤ **Today**: - Subcritical Hopf bifurcations :(
- Scaling of equations
- Center manifold theory

➤ Subcritical Hopf Bifurcations

➤ Eg: 1 $\left.\begin{array}{l}\dot{r} = \alpha r + r^3 - r^5 \\ \dot{\theta} = 1\end{array}\right\}$ See Khalil for $\dot{x}_1 = \ldots$, $\dot{x}_2 = \ldots$

$\Rightarrow \dot{r} = \alpha r + r^3 - r^5 = f(r)$

Sol^{n}s of $f(\bar{r}) = 0 \Rightarrow -\bar{r}(\bar{r}^4 - \bar{r}^2 - \alpha) = 0$

$\underbrace{\bar{r} = 0}_{\text{Equ}^m\text{ point}}$

$\bar{r}^4 - \bar{r}^2 - \alpha = 0$

let $\bar{q} = \bar{r}^2 \Rightarrow \bar{q}^2 - \bar{q} - \alpha = 0$

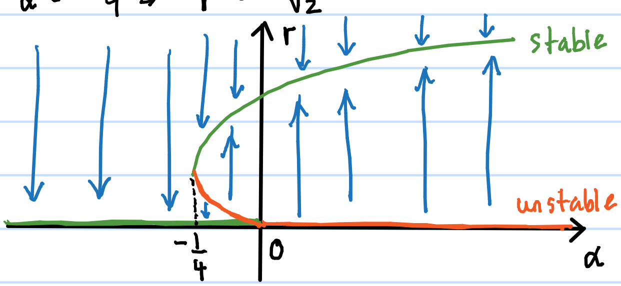
$$\bar{q}_{1,2} = \frac{1 \pm \sqrt{1^2 - 4(1)(-\alpha)}}{2} = \frac{1}{2} \pm \frac{\sqrt{1+4\alpha}}{2}$$

$\bar{r}^2 \rightarrow$ sol^n only if $\bar{q}_{1,2} \in \mathbb{R}$

$1 + 4\alpha \geq 0$

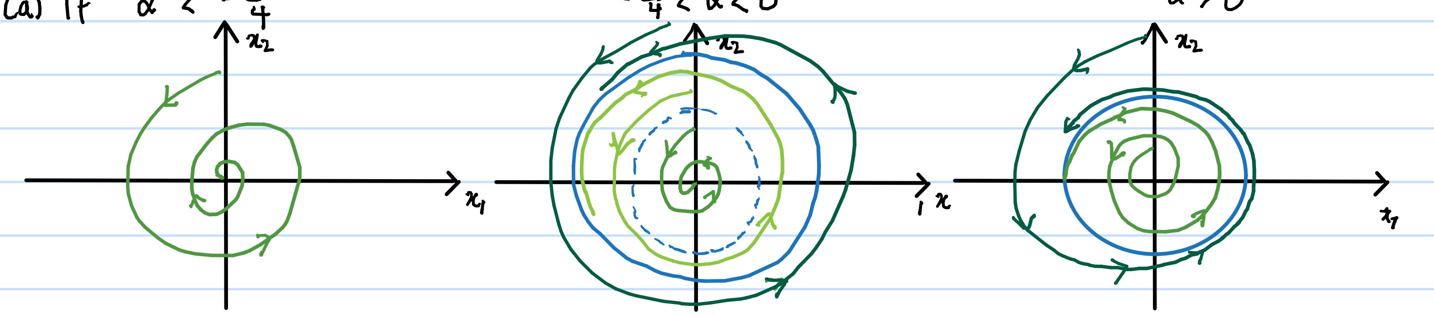
$\alpha \geq -\frac{1}{4}$

$\alpha = -\frac{1}{4} \Rightarrow \bar{r} = \frac{1}{\sqrt{2}}$



In the phase-plane:

(a) If $\alpha < -\frac{1}{4}$ | $-\frac{1}{4} < \alpha < 0$ | $\alpha > 0$



Bad ∵ always goes to a limit cycle → eg: @ 70mph, no matter where you start, end up @ 70mph - ticket!!

Scaling of Equations (Non-dimensional form of equations)

Eg ②: $\dot{x}_1 = -\alpha x_1 + \beta x_2$

$$\dot{x}_2 = \frac{\gamma x_1}{\delta + x_1^2} - \eta x_2$$

let $z_1 = \frac{x_1}{X_1}$, $z_2 = \frac{x_2}{X_2}$, $\tau = \frac{t}{T}$ | non-dimensional

Objective: write eqⁿs $\frac{dz}{d\tau} = f(z)$

$$\frac{dz_1}{d\tau} = \frac{1}{X_1}\frac{dx_1}{d\tau} = \frac{1}{X_1}\frac{dt}{d\tau}\cdot\frac{dx_1}{dt} = \frac{T}{X_1}\left(-\alpha x_1 + \beta x_2\right) = \frac{T}{X_1}\left(-\alpha X_1 z_1 + \beta X_2 z_2\right)$$

$$\frac{dz_2}{d\tau} = \frac{1}{X_2}\frac{dx_2}{d\tau} = \frac{1}{X_2}\frac{dt}{d\tau}\cdot\frac{dx_2}{d\tau} = \frac{T}{X_2}\left(\frac{\gamma x_1}{\delta + x_1^2} - \eta x_2\right) = \frac{T}{X_2}\left(\frac{\gamma X_1 z_1}{\delta + X_1^2 z_1^2} - \eta X_2 z_2\right)$$

If X_i and T are properly selected, can bring system to the form:

$$\frac{dz_1}{d\tau} = -az_1 + z_2 \qquad \frac{dz_2}{d\tau} = \frac{z_1}{1+z_1^2} - bz_2$$

(down to only two parameters: a & b)

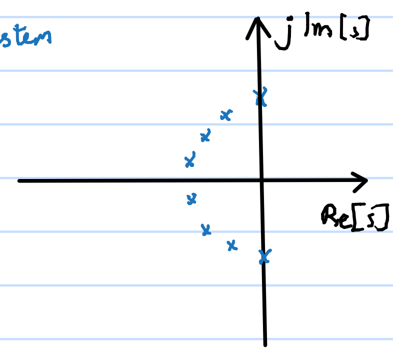
Center Manifold Theory

- Nonlinear system: $\dot{x} = f(x)$ —— (I) $\quad x(t) \in \mathbb{R}^n$ ← order of system

  let $f(0) = 0 \Rightarrow \bar{x} = 0$ is an equᵐ pt.

  let $A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}$ have k eigenvalues on the $j\omega$ axis, $n-k$ eigenvalues in the LHP

  ↳ Linearisation not helpful



- Challenge: cannot say anything about $\bar{x} = 0$ using linearisation

Write (I) as: $\quad \dot{x} = Ax + \tilde{f}(x) \quad$ (possible ✓)

- If we use Taylor series around $\bar{x} = 0$

$$f(x) = \cancel{f(0)}^{\,0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}}_{A}\cdot x + \text{h.o.t}$$

$$\tilde{f}(x) = f(x) - \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}\cdot x$$

Properties of $\tilde{f}$: $\quad \tilde{f}(0) = f(0) - \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}\cdot 0 = 0$

$$\frac{\partial \tilde{f}}{\partial x} = \frac{\partial f}{\partial x} - \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} \Rightarrow \left.\frac{\partial \tilde{f}}{\partial x}\right|_{\bar{x}=0} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} - \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} = 0$$

Summary: Can write (I) as:

$$\boxed{\begin{aligned} &\dot{x} = Ax + \tilde{f}(x) \\ &\tilde{f}(x) = 0 \\ &\left.\frac{\partial \tilde{f}}{\partial x}\right|_{\bar{x}=0} = 0 \end{aligned}}$$

Introduce change of coordinates:

$$\begin{array}{l} \mathbb{R}^k \leftarrow \\ \mathbb{R}^{n-k} \leftarrow \end{array} \begin{bmatrix} y \\ z \end{bmatrix} = Tx \quad \text{such that} \quad \left.\begin{aligned} \dot{y} &= A_1 y + g_1(y,z) \\ \dot{z} &= A_2 z + g_2(y,z) \end{aligned}\right\} ⊛ \quad \left| \begin{aligned} & g_i(0,0) = 0 \\ & \frac{\partial g_i}{\partial y}\Big|_{(0,0)} = \frac{\partial g_i}{\partial z}\Big|_{(0,0)} = 0 \end{aligned} \right.$$

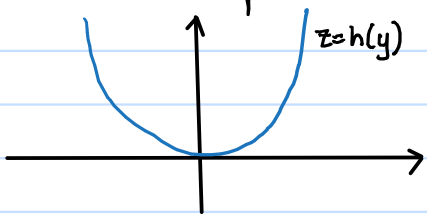
$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} g_1(y,z) \\ g_2(y,z) \end{bmatrix}$$

where A_1 contains eigenvalues on the $j\omega$ axis

A_2 contains eigenvalues in the LHP

➤ **Thm:** There is an invariant manifold $z = h(y)$ in the neighbourhood of the origin that satisfies $h(0) = 0$ and $\frac{\partial h}{\partial y}\Big|_0 = 0$

Geometric interpretation:



- invariant ~ start there, stay there
- manifold : surface in a higher dimensional state

Study 1^{st} eq^n (reduced-order system) w/ $z = h(y)$, then stability properties of the reduced-order system determines stability properties of the whole system.

Main result ↓

➤ **Thm:** If the origin of the reduced system $\dot{y} = A_1 y + g_1(y, h(y))$ is asymptotically stable (or unstable) then: the origin of ⊛ is asymptotically stable (or unstable)

Key challenge: Characterise $h(y)$ ← center manifold

Introduce $w = z - h(y)$

If $w \equiv 0 \Rightarrow \dot{w} \equiv 0$ (invariant manifold)

$$\dot{w} = \dot{z} - \dot{h} = \dot{z} - \frac{\partial h}{\partial y}\dot{y} = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\left[A_1 y + g_1(y, h(y))\right] = 0$$

Eq^n that characterises a manifold:

$$\boxed{A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\left[A_1 y + g_1(y, h(y))\right] = 0} \quad \text{— (II)}$$

➤ **Eg ③:** Let $y(t) \in \mathbb{R}$ and look for approximate solutions of (II)

Use Taylor series of $h(y)$ around zero:

$$h(y) = \cancel{h(0)}^{0} + \cancel{\frac{\partial h}{\partial y}\Big|_0}^{0} y + h_2 y^2 + h_3 y^3 + O(y^4)$$

from earlier.

$h_2 \to \frac{\partial^2 h}{\partial y^2}\Big|_0$, $h_3 \to \frac{\partial^3 h}{\partial y^3}\Big|_0$

locally, only quadratic + h.o.t

∵ already captured linear terms

$$\Rightarrow \quad h(y) = h_2 y^2 + h_3 y^3 + O(h^4)$$

Eg 4 :

$$\dot{y} = 0 \cdot y + \underbrace{yz}_{g_1(z)}$$
$$\dot{z} = -z + ay^2$$

$A_1 = 0$, $A_2 = -1$,

Plug $g = h(y) = h_2 y^2 + h_3 y^3 + O(h^4)$ in and equate equal orders in y.

turns out $h_2 = a \rightarrow a > 0$

$a < 0$