EE 8215 : Nonlinear Systems

Lecture 9 : Feb 18^{th} 2016, Thu

▶ Last time: - Existence & uniqueness
- Continuous dependence on ICs & parameters
- Lipschitz continuity

▶ Today: - Sensitivity w.r.t parameters
- Sensitivity equations
- Lyapunov-based stability (if time permits)

$\Rightarrow$ • $\dot{x} = f(x, \mu, t)$ —— (1) $x(t) \in \mathbb{R}^n$: state vector , $\mu(t) \in \mathbb{R}^m$: vector of parameters

• Assume f is continuous w.r.t both x and μ; f is continuously differentiable (i.e. solution exists — at least on a finite time interval.)

• let $\bar{\mu} \in \mathbb{R}^m$ be a fixed vector of parameters

Q> What happens if we perturb $\bar{\mu}$?

→ Differentiate f w.r.t vector of parameters & look at resulting ODE

let $x(t, \bar{\mu})$ denote the solution for $\bar{\mu}$

$$x(t,\mu) = x(t,\bar{\mu}) + \underbrace{\left.\frac{\partial x}{\partial \mu}\right|_{\bar{\mu}}}_{S(t)\ \rightarrow\ \text{"sensitivity" matrix}} (\mu - \bar{\mu}) + \text{h.o.t.}$$

$$\approx x(t+\mu) + S(t)(\mu - \bar{\mu})$$

Can figure out what solution does w.r.t $\bar{\mu}$ by looking at sensitivity matrix

→ Objective: Find the eq^n that governs the evolution of:

$$S(t) = \left.\frac{\partial x(t,\mu)}{\partial \mu}\right|_{\bar{\mu}} = x_\mu(t, \bar{\mu})$$

$$x(t,\mu) = x_0 + \int_{t_0}^{t} f(x(\tau,\mu), \mu, \tau)\, d\tau$$

Plan of action: • Differentiate w.r.t. μ first, then w.r.t. time

↳ Gives differential eq^n for $S(t)$

$$x_\mu(t,\mu) = \cancelto{0}{\frac{\partial x_0}{\partial \mu}} + \int_{t_0}^{t} \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \mu} + \frac{\partial f}{\partial \mu} \right) d\tau$$

$$x_\mu(t,\mu) = \int_{t_0}^{t} \left[\frac{\partial f(x(\tau,\mu),\mu,\tau)}{\partial x} \cdot x_\mu(\tau,\mu) + f_\mu(x(\tau,\mu),\mu,\tau) \right] d\tau \quad \forall \mu$$

- Evaluate at $\mu = \bar{\mu}$:

$$x_\mu(t,\bar{\mu}) = S(t) = \int_{t_0}^{t} \frac{\partial f(x(\tau,\bar{\mu}),\bar{\mu},\tau)}{\partial x} S(\tau) + f_\mu(x(\tau,\bar{\mu}),\bar{\mu},\tau) \,]\, d\tau$$

- Differentiate w.r.t. time :

$$\dot{S}(t) = \mathcal{A}(t) \cdot S(t) + \mathcal{B}(t)$$

→

where $\mathcal{A}(t) = \dfrac{\partial f(x(t,\bar{\mu}),\bar{\mu},t)}{\partial x}$ and $\mathcal{B}(t) = \dfrac{\partial f(x(t,\bar{\mu}),\bar{\mu},t)}{\partial \mu}$

Note: both matrices depend on solution $x(t,\bar{\mu})$

$$\dot{x} = f(x,\bar{\mu},t)$$
$$\dot{S} = \mathcal{A}(t)\, S(t) + \mathcal{B}(t)$$

$\Downarrow$ one-way coupling

↳ Simulate → difficult to derive analytical solⁿ.
$\mathcal{A}(t)$ & $\mathcal{B}(t)$ are functions of $x(t,\bar{\mu})$

➤ Eg ①: Fold bifurcation: $\dot{x} = x^2 + \mu$

$f(x,\mu) = x^2 + \mu$

$\dfrac{\partial f}{\partial x} = 2x$, $\dfrac{\partial f}{\partial \mu} = 1$

$$\dot{x} = x^2 + \bar{\mu}\,; \qquad x(0) = x_0$$
$$S = 2\underbrace{x(t)}\, S + 1\,; \qquad S(0) = 0$$

↳ Fixed trajectory that starts at x_0 for fixed $\bar{\mu}$

➤ Eg ②: (Khalil Eg. 3.17)

$$\dot{x}_1 = x_2 \qquad = f_1$$
$$\dot{x}_2 = -c\sin(x_1) - [a + b\cos(x_1)]x_2 \qquad = f_2$$

$\mu = [a, b, c]^T \in \mathbb{R}^3$ $\quad x(t) \in \mathbb{R}^2$ $\quad$ Given $\bar{\mu} = [1\ 0\ 1]^T$

$$S = \left.\begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} \end{bmatrix}\right|_{\mu=\bar{\mu}} \in \mathbb{R}^{2\times 3}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c\cdot\cos(x_1) + b x_2 \sin(x_1) & -(a+b\cos(x_1)) \end{bmatrix}$$

$$\mathcal{A}(t) = \begin{bmatrix} 0 & 1 \\ -\cos(x_1(t)) & -1 \end{bmatrix}$$

$$\mathcal{B}(t) = \begin{bmatrix} 0 & 0 & 0 \\ -x_2(t) & -x_2(t)\cos(x_1(t)) & -\sin(x_1(t)) \end{bmatrix}$$

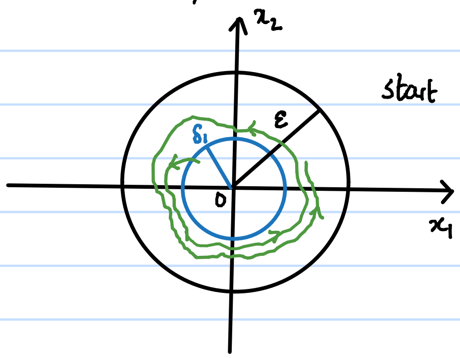
➤ Lyapunov-based Stability

- stability w.r.t. initial conditions
- natural : BIBO → w.r.t. inputs (undergrad)
- most commonly used notion of stability in science & engineering

- Consider: $\dot{x} = f(x)$ → time-invariant $\qquad x(t) \in \mathbb{R}^n$
- Assume $f(0) = 0 \Rightarrow \bar{x} = 0$ is an equ^m pt.
  (w/o loss of generality) → if not ⇒ change coordinates
  ↳ Consider $\dot{x} = f(x)$ w/ $f(\bar{x}) = 0$ w/ $\bar{x} \neq 0$
  let $z(t) = x(t) - \bar{x} \Rightarrow \dot{z} = \dot{x} - \cancel{\dot{\bar{x}}}^{\to 0} = f(x)$
  $= f(z + \bar{x})$
  Since $f(\bar{x}) = 0 \Rightarrow \bar{z} = 0$ is an equ^m pt.

- Stability: perturb equilibrium point w/ ICs and study qualitative behaviour of the resulting trajectories.

1) $\bar{x} = 0$ is stable (in the sense of Lyapunov)
   if $\forall\ \varepsilon > 0,\ \exists\ \delta_1 > 0$
   s.t. $\|x_0\| < \delta_1 \Rightarrow \|x(t, x_0)\| < \varepsilon$
   for all times
   if $\bar{x} = 0$ not an equ^m pt: $\|x_0 - \bar{x}\| < \delta_1 \Rightarrow \|x(t, x_0) - \bar{x}\| < \varepsilon$
   i.e. you start close, you stay close ...



start at δ_1 stay w/in ε

eg: room temperature $\sim 69°F$

2) Unstable if it is not stable

3) Locally asymptotically stable (LAS):
   - if it is stable
   - if $\exists\ \delta_2 > 0$ s.t. $\forall\ \|x_0\| < \delta_2 \Rightarrow \lim_{t \to \infty} \|x(t, x_0)\| = 0$
     (attractiveness)
     ↳ doesn't say anything about stability

4) Globally asymptotically stable (GAS)
   - if 3) holds for any δ_2