Linear Algebra Homework

Problem 1 If $A \in R^{n \times n}$ and α is a scalar what is $det(\alpha A)$? What is $det(-A)$?

Problem 2 If A is orthogonal, what is $det(A)$? If A is unitary what is $det(A)$?

Problem 3 Let $x, y \in R^n$. Show that $det(I - xy^T) = 1 - y^T x$.

Problem 4 Show that the product of orthogonal matrices is orthogonal.

Problem 5 The *trace* of a matrix $A \in R^{n \times n}$ is defined as the sum of the diagonal elements:

$$Tr(A) = \sum_{i=1}^{n} a_{ii}.$$

1. Show that the trace of a matrix is a linear function that is if $A, B \in R^{n \times n}$ and $\alpha, \beta \in R$ then $Tr(\alpha A + \beta B) = \alpha Tr(A) + \beta Tr(B)$.
2. Show that $Tr(AB) = Tr(BA)$ eventhough in general $AB \neq BA$.

Problem 6 Suppose $\{w_1, w_2, \ldots, w_n\}$ is a linearly dependent set. Then show that one of the vectors must be a linear combination of the others.

Problem 7 Let $x_1, x_2, \ldots, x_k \in R^n$ be mutually orthogonal vectors. Show that $\{x_1, x_2, \ldots, x_k\}$ must be a linearly independent set.

Problem 8 Let $w_1, w_2, \ldots, w_n$ be a set of orthonormal vectors in R^n. Show that $Aw_1, \ldots, Aw_n$ are also orthonormal if and only if $A \in R^{n \times n}$ is orthogonal.

Problem 9 Consider vectors $v_1 = (2\ 1)^T$ and $v_2 = (3\ 1)^T$. Show that v_1 and v_2 form a basis for R^2. Find the coordinates of $v = (4\ 1)^T$ in this basis. What are the coordinates of v in the basis $e_1 = (1\ 0)^T$ and $e_2 = (0\ 1)^T$.

Problem 10 Let $\mathcal{P}$ denote the set of polynomials of degree less than or equal to two of the form $p_0 + p_1 x + p_2 x^2$ where p_0, p_1 and $p_2 \in R$. Show that $\mathcal{P}$ is a vector space with the reals as the scalars. Show that the polynomials $1, x,$ and $2x^2 - 1$ are a basis for $\mathcal{P}$. Find the components of the polynomial $2 + 3x + 4x^2$ with respect to this basis.

Problem 11 Suppose V and W are subspaces of X. Then show that

1. $V + W$ and $V \cap W$ are subspaces of X.

2. If $V \oplus W = X$ then show that every $x \in X$ can be written uniquely in the form $x = v + w$ where $v \in V$ and $w \in W$.
3. $dim(V + W) = \dim(V) + dim(W) - dim(V \cap W)$.

Problem 12 Let X be the vector space of of $n \times n$ matrices with reals as the scalars. Let V be the set of $n \times n$ skew symmetric matrices (a matrix A is skew symmetric if $A = -A^T$) and let W be the set of $n \times n$ symmetric matrices. Show that $X = V \oplus W$.

Problem 13 Let $\mathcal{A} : V \to W$ denote the operator from the vector space V of polynomials of degrees less than or equal to n to the vector space W which is the same vector space as W with $\mathcal{A}$ defined by

$$\mathcal{A}v = \frac{d^k v(t)}{dt^k}.$$

Show that A is a linear operator. Find the null space and range space of $\mathcal{A}$.

Problem 14 Let V be the space of all polynomials. Let $\mathcal{A}$ be defined by $\mathcal{A}v^2$. Show that $\mathcal{A}$ is not linear.

Problem 15 Let v_1, v_2, v_3 be a basis for the vector space V and let $\mathcal{A} : V \to V$ be a linear operator such that

$$\mathcal{A}v_1 = v_1 + v_2;\ \mathcal{A}v_2 = v_3 \text{ and } \mathcal{A}v_3 = v_1 - v_2.$$

Find the matrix representation of $\mathcal{A}$ with respect to this basis. Let $\hat{v}_1, \dots, \hat{v}_3$ be another basis for V related to the basis of the previous problem by

$$\hat{v}_1 = v_1 - v_2; \hat{v}_2 = v_1 + v_2; \hat{v}_3 = v_1 - v_3.$$

Obtain the matrix representation with respect to the new basis. Also, find the change of basis matrix.

Problem 16 Let $v_1, \dots, v_n$ form a basis for a vector space V. Let $Q = (q_{ij}) \in R^{n \times n}$ be a nonsingular matrix. Show that

$$\hat{v}_j := \sum_{i=1}^{n} q_{ij} v_i,\ j = 1, \dots, n$$

also forms a basis for V.

Problem 17 Let V and W be a vector space and let $[V, W]$ represent the collection of all linear operators from V to W. Show that $[V, W]$ is a vector space and $dim([V, W]) = nm$ if $dim(V) = n$ and $dim(W) = m$.

Problem 18 Determine the bases for the range and null spaces of the linear operator $\mathcal{A} : R^3 \to R^2$ defined by

$$\mathcal{A}v := \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

where $v = (\alpha_1, \alpha_2, \alpha_3)^T$.

Problem 19 Let A be a $m \times n$ matrix, b be a $m-$ vector and (A, b) be the $m \times (n+1)$ matrix with the additional column b. how that $b \in range(A)$ if and only if $rank(A) = rank(A, b)$.

Problem 20 Let λ and v be a an eigenvalue and a corresponding eigenvector of a linear operator $\mathcal{A} : V \to V$ and let $v_1, \ldots, v_n$ be a basis for V. Let A be the matrix representation of $\mathcal{A}$ in the given basis. Show that λ and α are an eigenvalue and eigenvector of A where α is the coordinate vector of v.

Problem 21 Let A be a $m \times n$ matrix. Prove that

$$dim(range(A^*)) + dim(null(A^*)) = m.$$

Problem 22 Let A and P be $n \times n$ matrices with P nonsingular. Show that $trace(P^{-1}AP) = trace(P)$.

Problem 22 Let H be an $n \times n$ Hermitian positive definite matrix and A an $n \times m$ matrix. Show that $rank(A) = rank(A^*HA)$.

Problem 23 Prove that if A is a $m \times n$ matrix then $null(A^*A) = null(A)$ and $rank(A^*A) = rank(A)$.

Problem 24 Prove that if $A \in R^{m \times n}$ then $rank(A) =$ number of independent columns of $A =$ number of independent rows of A.