Feedback

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Single Input and Single output Interconnections:

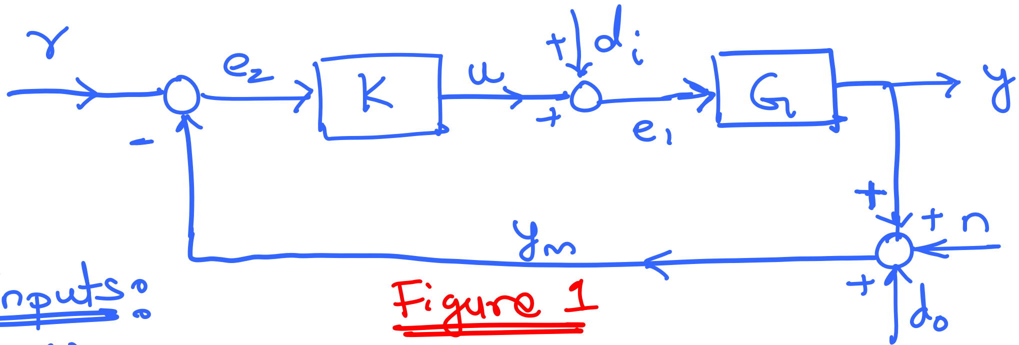

Figure 1

Inputs:

- $r(t)$ is the reference input.
  - A typical requirement is that the error $e = r - y$ is small
- $d_i(t)$ is the input disturbance to the plant G.
- n is the noise that models the measurement noise (of measuring y; the output of the plant G).
- d_o is the output disturbance

Internal variables

e_2, u, e_1, y, y_m

CLOSED LOOP MAPS

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→ A transfer function from any particular input to any particular output can be found by setting all other inputs to zero

Transfer function with input r

we set $d_0 = n = d_i = 0$

(a) Transfer function from r to e_2

It follows that

$$e_2 = r - y$$

and

$$y = GKe_2$$

$$\Rightarrow \quad e_2 = r - GKe_2$$

$$\Rightarrow \quad (1+GK)e_2 = r$$

$$\Rightarrow \quad \frac{e_2}{r} = \frac{1}{1+GK}$$

(b) Transfer function from r to u

- $u = Ke_2 = K\frac{1}{1+GK}r$

$$\Rightarrow \quad \frac{u}{r} = \frac{K}{1+GK}.$$

Input r

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(c) Transfer function from r to e_1

as $d_i = 0$; $u = e_1$ and

$$\frac{e_1}{r} = \frac{K}{1+GK}.$$

(d) Transfer function from r to y

$$y = GKe_2$$

$$\Rightarrow y = GK\frac{1}{1+GK}r$$

$$\Rightarrow \frac{y}{r} = \frac{GK}{1+GK}$$

(e) Transfer function from r to y_m

with $n = d_o = 0$ $y_m = y$

and

$$\frac{y_m}{r} = \frac{GK}{1+GK}.$$

Input d_i

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Transfer function with input d_i

Set $r = n = d_o = 0$

(a) output e_1:

$$e_1 = d_i + u$$
$$= d_i + K e_2$$
$$= d_i + K(-y)$$
$$= d_i - KGe_1$$

$$\Rightarrow \quad \frac{e_1}{d_i} = \frac{1}{1+KG}$$

(b) output y and y_m

$$\frac{y}{d_i} = \frac{y_m}{d_i} = \frac{G}{1+KG}$$

(c) output e_2

$$\frac{e_2}{d_i} = -\frac{y_m}{d_i} = -\frac{G}{1+KG}.$$

(d) output u

$$\frac{u}{d_i} = \frac{Ke_2}{d_i} = -\frac{KG}{1+KG}.$$

$$\frac{u}{d_i} = \frac{K e_2}{d_i} = -\frac{KG}{1+KG}.$$

Input n

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Transfer functions with input n:

Set $r = d_i = d_o = 0$

(a) output e_2:

$$\frac{e_2}{n} = -\frac{1}{1+GK}$$

(b) output u

$$\frac{u}{n} = -\frac{K}{1+GK}$$

(c) output y

$$\frac{y}{n} = -\frac{GK}{1+GK}$$

(d) output y_m

$$y_m = y + n = \left(1 - \frac{GK}{1+GK}\right)n = \frac{1}{1+GK}n$$

$$y_m/n = 1/1+GK.$$

Summary of tfs

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- Essentially. the ~~the~~ complete set of transfer functions are

① $\frac{1}{1+GK}$; ② $\frac{G}{1+GK}$; ③ $\frac{K}{1+GK}$ and ④ $\frac{GK}{1+GK}$.

Thus, the above interconnection is Bounded input and bounded output stable if and only if

$$\frac{1}{1+GK},\quad \frac{G}{1+GK},\quad \frac{K}{1+GK} \text{ and } \frac{GK}{1+GK}$$

are all stable. i.e. the poles of the these transfer functions are in the strict left half plane.

Imp. Closed-loop maps

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Consider the following block diagram

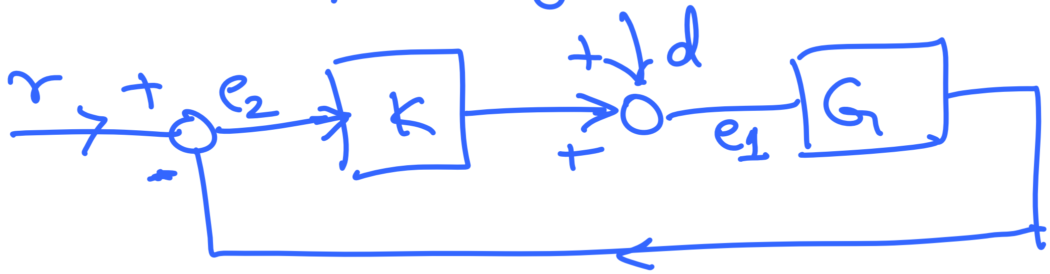

Figure 2:

Then

$$e_1 = d + Ke_2$$
$$e_2 = r - Ge_1$$

$$\Rightarrow \quad e_1 = d + K(r - Ge_1)$$

$$\Rightarrow \quad e_1(1 + KG) = d + Kr$$

$$\Rightarrow \quad e_1 = \frac{1}{1+KG} d + \frac{K}{1+KG} r$$

and

$$e_2 = r - Ge_1$$
$$= r - \frac{Gd}{1+KG} - \frac{KG}{1+KG} r$$
$$= \frac{1}{1+KG} r - \frac{G}{1+KG} d$$

Stability Equivalence

Transfer functions are

$$\frac{1}{1+KG};\ \frac{K}{1+KG},\ \frac{G}{1+KG}$$

and thus the above interconnection is stable if

$$\frac{1}{1+KG},\ \frac{K}{1+KG} \text{ and } \frac{G}{1+KG} \text{ have}$$

all the poles in the strict left half plane.

Note that

$$\frac{GK}{1+KG} = 1 - \frac{1}{1+KG}$$

and thus, if $\frac{1}{1+GK}$ is stable then so is $\frac{GK}{1+KG}$.

Thus, the interconnection in Figure 1 is bounded input bounded output stable if and only if Figure 2 is bounded input bounded output stable (with inputs r and d and outputs e_1 and e_2).

Stability Equivalence

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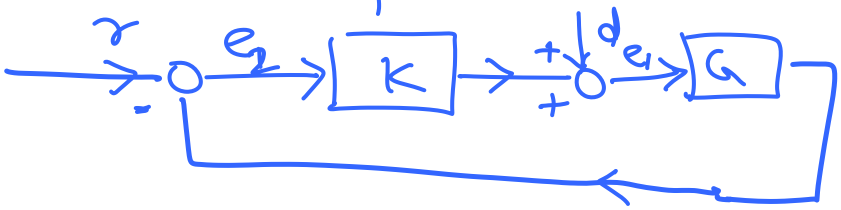
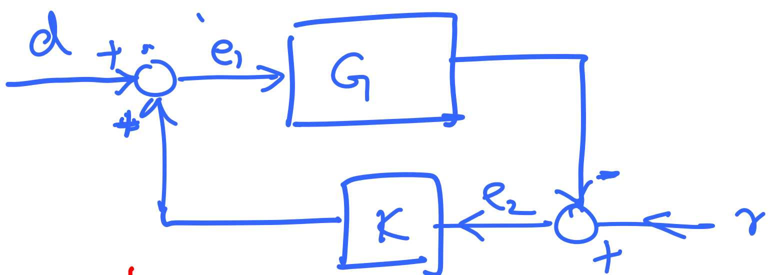
Thus we will focus on



$\equiv$



Thus we have

Theorem 1:

Bounded input Bounded stability if and only if

$\frac{G}{1+GK}$, $\frac{K}{1+GK}$ and $\frac{1}{1+GK}$ are stable

transfer functions

② Let $G = \frac{n_G}{d_G}$; $K = \frac{n_K}{d_K}$, where n_G, d_G, n_K, d_K are all polynomials.

(We assume that n_G and d_G have no common roots

and we assume n_K and d_K have no common roots)

Example

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- Note that assuming that n_G, d_G have no common roots and n_K, d_K have no common roots does not imply

$n_G n_K$ and $d_G d_K$ have no common roots

Example: let $K = \frac{(s-1)}{(s+3)(s+2)}$; $G = \frac{(s+3)}{(s+2)(s-1)}$

$n_K = s-1$; $d_K = (s+3)(s+2)$

$n_G = (s+3)$; $d_G = (s+2)(s-1)$

and

$n_G n_K = (s-1)(s+3)$

$d_G d_K = (s+3)(s+2)(s-1)$

Thus, $d_G d_K$ and $n_G n_K$ have two common terms $(s+3)$ and $(s-1)$.

Stability theorem

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Theorem 2:

The interconnection shown in Figure 1 and Figure 2 are bounded input bounded output stable if the polynomial $d_G d_K + n_G n_K$ (where $G = \frac{n_G}{d_G}$ and $K = \frac{n_K}{d_K}$) has all roots in the strict left half plane.

Proof: Note that from the previous theorem Figure 1 and Figure 2 are BIBO stable

- if and only if

$$\frac{1}{1+GK} = \frac{1}{1+\frac{n_G n_K}{d_G d_K}} = \frac{d_G d_K}{d_G d_K + n_G n_K}$$

$$\frac{G}{1+GK} = \frac{n_G d_K}{d_G d_K + n_G n_K}$$

$$\frac{K}{1+GK} = \frac{n_K d_G}{d_G d_K + n_G n_K}$$

have all poles in the strict left half plane

Clearly if $d_G d_K + n_G n_K$ has no roots in the strict left half plane then

Example

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all the above transfer functions will have no poles in the right half plane and BIBO stability follows.

$\square$

<u>Remark</u>: Note that it is possible that

$$\frac{1}{1+GK} \text{ is stable}$$

without $\frac{G}{1+GK}$ or $\frac{K}{1+GK}$ being stable.

<u>Example</u>: let $G = \frac{1}{(s-1)}$; $K = \frac{(s-1)}{(s+2)}$

Thus;

$$1+GK = 1 + \frac{1}{s-1}\frac{(s-1)}{(s+2)}$$

$$= 1 + \frac{1}{s+2}$$

$$= \frac{s+3}{s+2}$$

$$\Rightarrow \frac{1}{1+GK} = \frac{s+2}{s+3} \text{ which is stable}$$

However $\frac{G}{1+GK} = \frac{1}{(s-1)}\frac{(s+2)}{(s+3)}$ which is not stable.

Pole zero Cancellation

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The problem is unstable pole-zero cancellation between G and K.

Note that G has a unstable pole at 1 and K has a unstable zero at the same location 1. Thus

$$GK = \frac{1}{(s-1)} \frac{(s-1)}{(s+2)} = \frac{1}{(s+2)}$$

and there is a cancellation of an unstable factor.

The following theorem holds:

Theorem 3: The interconnection in Figure 1 and Figure 2 are BIBO stable if and only if

(a) There are no unstable pole-zero cancellations when forming the product GK

(b) 1+GH has no zeros in the right half plane.

Proof.

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Proof: Note that with $G = \frac{n_G}{d_G}$; $K = \frac{n_K}{d_K}$

then
$$1 + GK = \frac{n_G n_K + d_G d_K}{d_G d_K}.$$

(1) Suppose (a) and (b) hold.

- Note that if $n_G n_K + d_G d_K$ has all roots in the strict left half plane the the system is BIBO stable (Theorem 2).

As (a) holds $(1+GK)$ has no zeros in the right half plane.

Thus, $\frac{n_G n_K + d_G d_K}{d_G d_K}$ has no zeros in the right half plane

Thus, $n_G n_K + d_G d_K$ can have zero in the right half plane only if such a factor is cancelled by the same factor of $d_G d_K$. i.e. there is a s_0 in the right half plane with $(n_G n_K + d_G d_K)(s_0) = 0$

$$(d_G d_K)(s_0) = 0$$

Charac. Polynomial

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but this implies that

$$0 = (n_G n_K)(s_0) + (d_G d_K)(s_0)$$
$$= (n_G n_K)(s_0) + 0$$
$$\Rightarrow (n_G n_K)(s_0) = 0$$

Thus, s_0 is such that $\underbrace{(n_G n_K)(s_0)}_{=0} = (d_G d_K)(s_0)$

and thus, there is a common factor $(s - s_0)$ between $n_G n_K$ and $d_G d_K$. Thus, there is a unstable pole-zero cancellation when the product $GK = \frac{n_G n_K}{d_G d_K}$ is formed and this is not allowed by (b).

Thus, if conditions (a) and (b) are met then the system in figure (1) and (2) are BIBO stable

□

Summary

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Summary :

For BIBO stability of Figure 1 ascertain that

(a) There is no unstable pole-zero cancellation in forming the product GK

(b) $1+GK$ has no zeros in the right half plane.

Equivalently ascertain that

Routh Hurwitz and see if polynomial
$d_G d_K + n_G n_K$
has any roots in the right halfplane.