Proportional-derivative Design.

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* The PD controller has the form

$$K(s) = K_P + K_D s$$ where K_P and K_D are constants

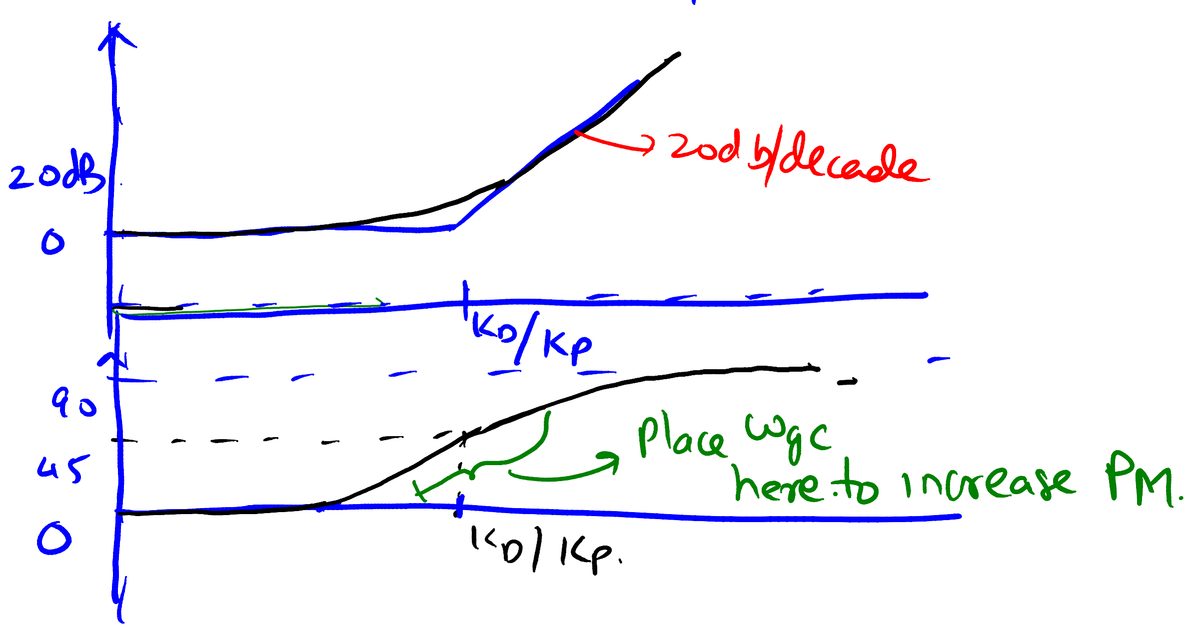
* Transforming to the standard form

$$K(s) = K_P \left[1 + \frac{s}{K_P/K_D} \right]$$

and thus, $K(s)$ has a break frequency at K_P/K_D.

$$|K(j\omega)| = 20\log K_P + 20\log \sqrt{1 + \left(\frac{K_D}{K_P}\omega\right)^2}$$

$$\angle K(j\omega) = \tan^{-1}\left(\frac{K_D \omega}{K_P}\right)$$



PD Design steps

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1. From Specifications obtain

   (a) PM_d = Phase Margin desired
   (b) ω_{gcd} = The gain crossover desired

2. From Bode plot of G obtain

   $PM_{have} = 180 + \angle G(j\omega_{gcd})$ the phase already present at gain crossover desired.

3. Determine ΔPM the deficit of phase margin to be provided by PD controller

   $$\Delta PM = PM_d - PM_{have}$$

4. Note that ΔPM has to be phase of the PD controller at ω_{gcd}. Thus

   $$\Delta PM = \angle K(j\omega_{gcd}) = \tan^{-1}\left(\frac{K_D}{K_P}\omega_{gcd}\right) \Rightarrow \frac{K_D}{K_P} = \frac{\tan(\Delta PM)}{\omega_{gcd}}.$$

(5.) Let $K_P = \dfrac{1}{\left|1 + \frac{K_D}{K_P} s\right| |G(s)|}$ at $s = j\omega_{gcd}$ which will place ω_{gcd} at K_P/K_D

PD design

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(6) $\frac{K_D}{K_P}$ is determined in step 4 and

K_P is determined in step 5

obtain $K_D = \left(\frac{K_D}{K_P}\right) K_P$

The PD controller is

$$K(s) = K_P + K_D s$$

Example

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$$G(s) = \frac{200}{s^2+4s+4}$$

with $K=1$ we have

$PM = 16$; $M_p = 64\%$

It is desired that a PD controller be designed to have the following specifications

(a) $M_p \le 16\%$

(b) ω_{gcd}; the desired gain crossover frequency of 14 rad/sec.

1. Thus, from (a) we have

$PM_d = 60$ degrees

$\omega_{gcd} = 14$ rad/se

2. $PM_{have} = 180 + \angle G(j\omega_{gcd}) = 16$ deg

3. $\Delta PM = PM_d - PM_{have} = \underset{+SM}{60} - 16 = 47$ degrees

4. $K_D/K_P = \tan(\Delta PM)/\omega_{gcd} = \frac{1}{14}\tan(47^\circ) = 0.0766$

5. $K_P = \frac{1}{\left|1+\frac{K_D}{K_P}s\right| \left|G(s)\right|_{s=j14}} = 0.682$

6. $K_D = (K_D/K_P)\,K_P = 0.0522$; $K(s) = 0.682 + 0.0522s$.

Bode plot of G(s)

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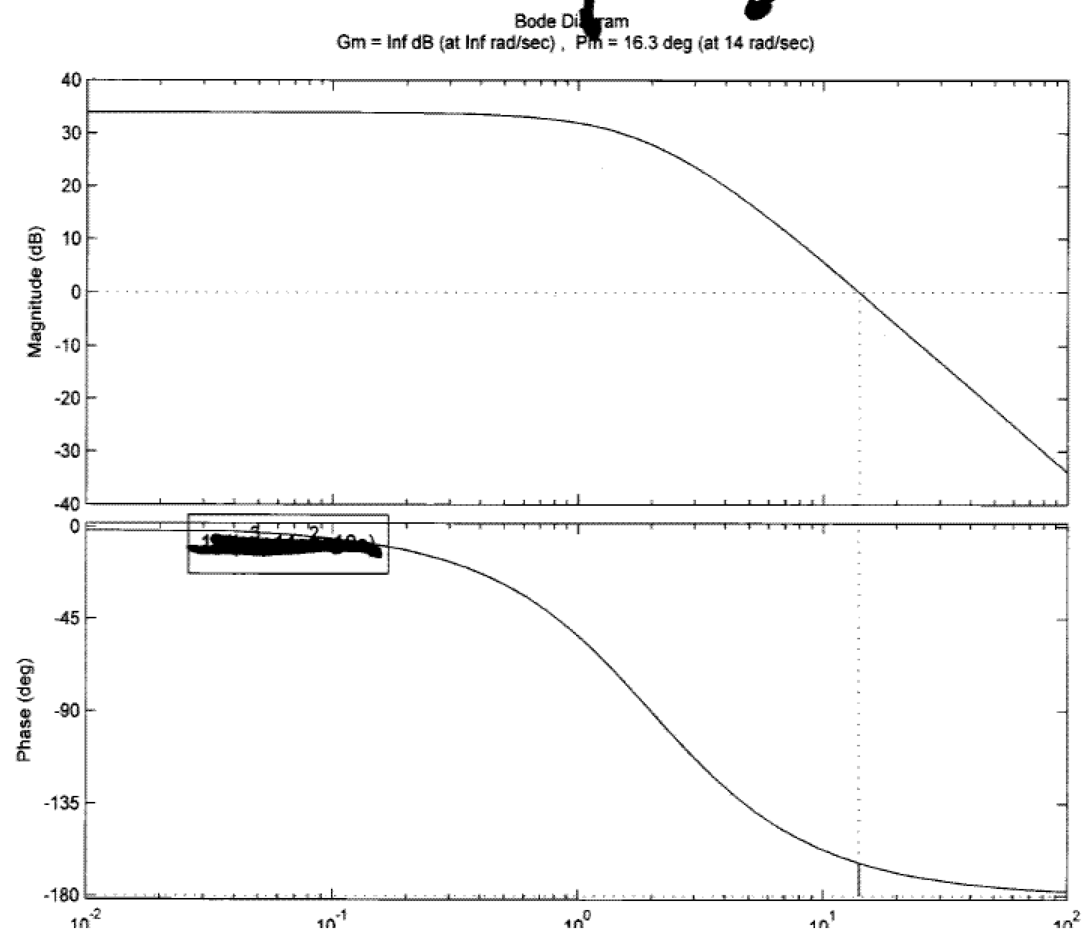
Bode plot of $\frac{200}{s^2+4s+4}$,

PM = 16.3°

$\omega_{gc} = 14$



Step response

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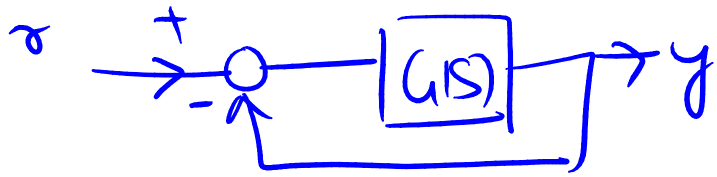

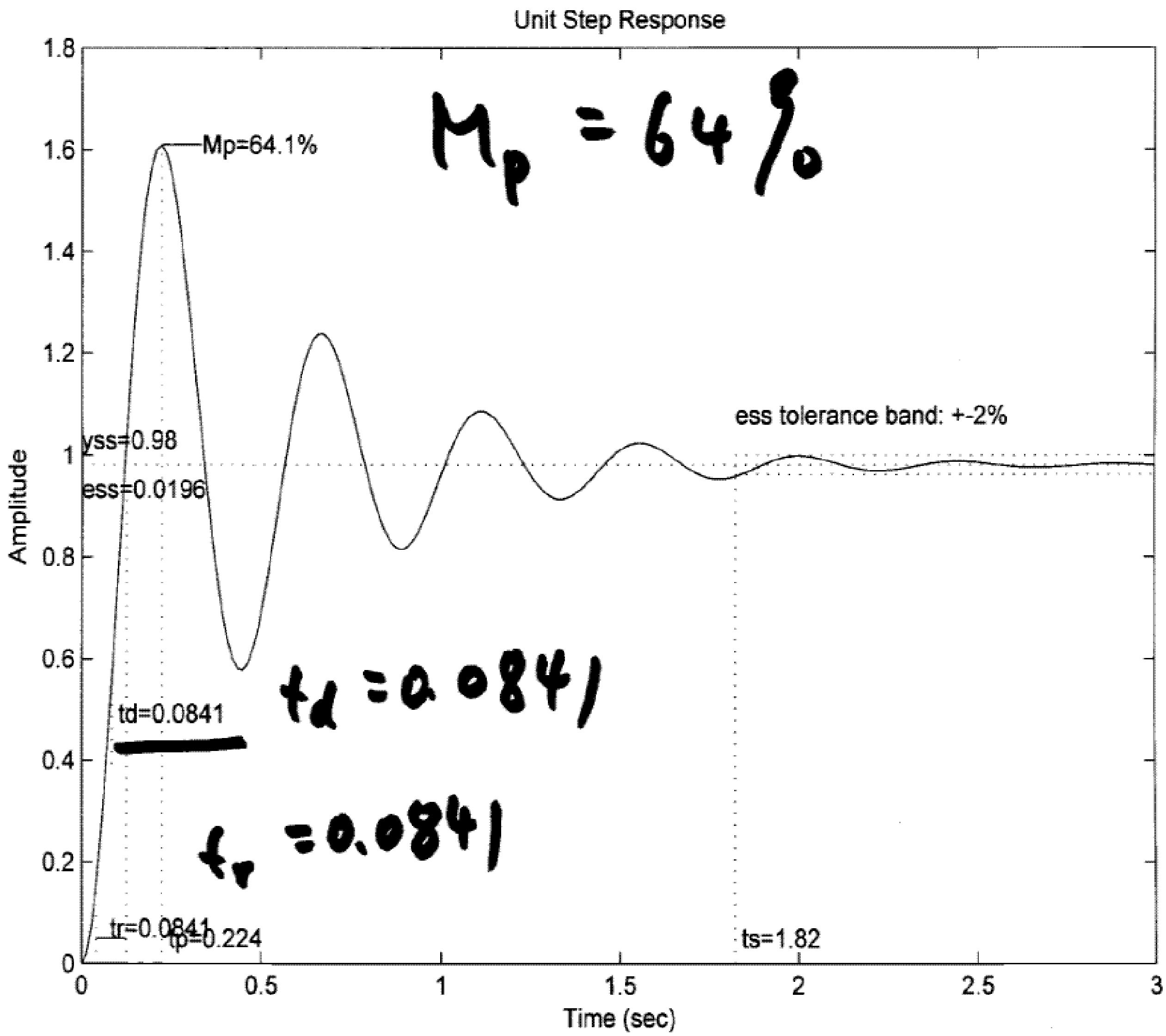

Bode of K(s)G(s)

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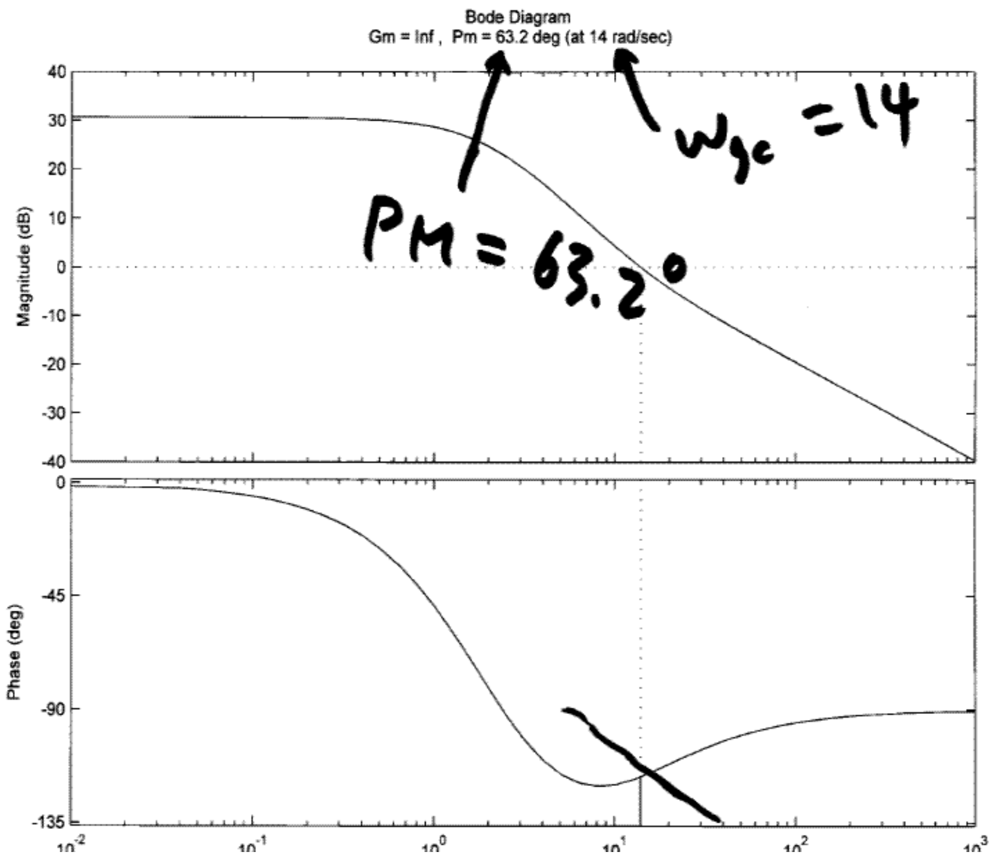
Bode plot of C(s)G(s)
with
$$\begin{cases} C(s) = 0.682 + 0.0522\,s \\ G(s) = 200/(s^2+4s+4) \end{cases}$$



Step response

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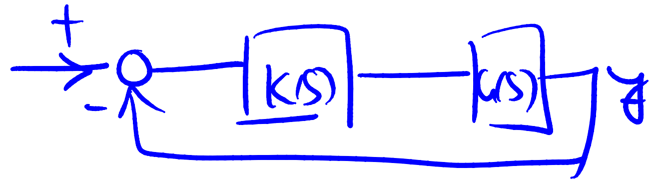
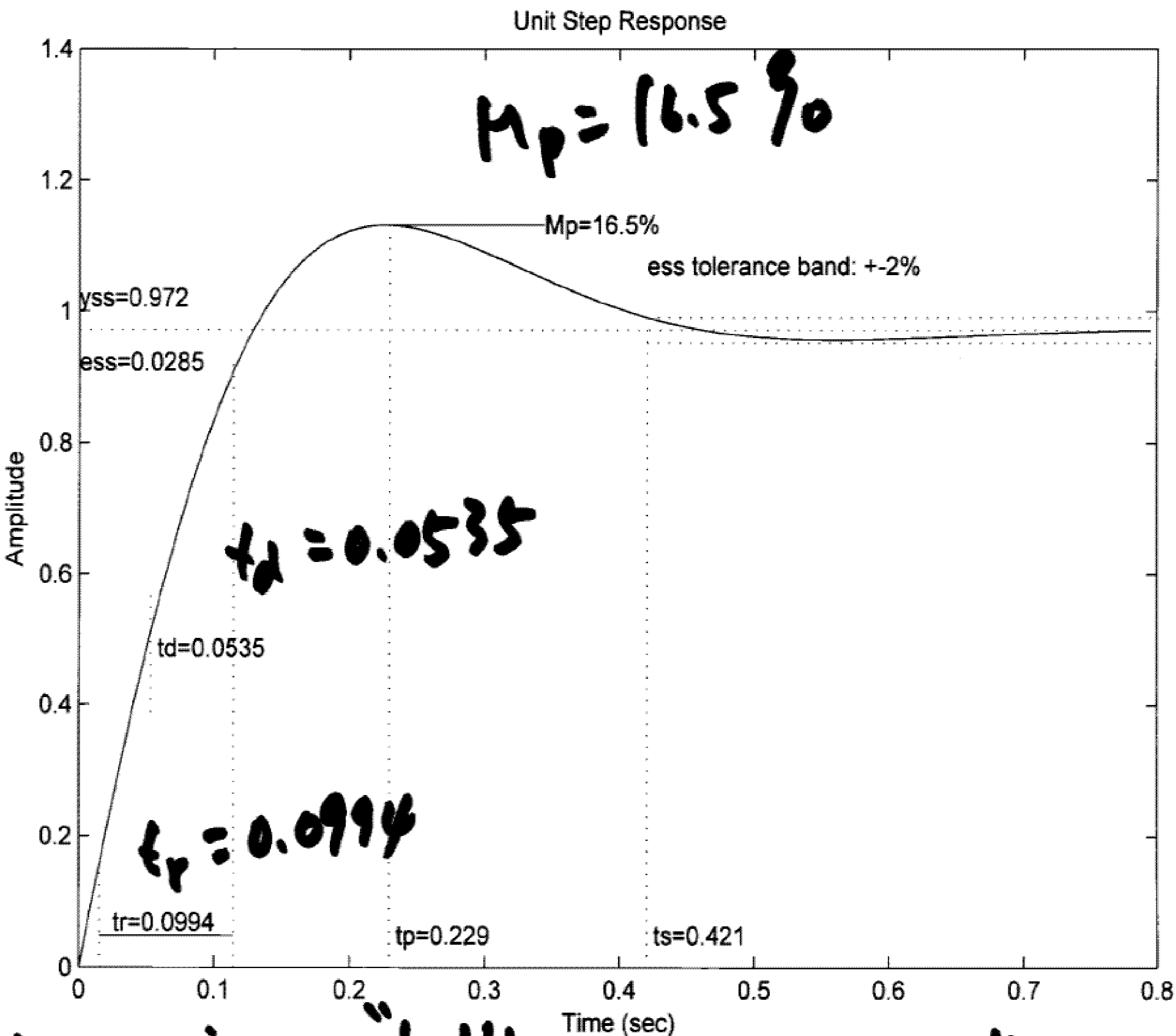


$K(S) = K_p + K_D S$



Mp is a "little too much" :)

Lead Controller

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$$G(s) = k \frac{Ts+1}{\alpha T s + 1}$$

$$G(j\omega) = k \frac{j\omega T + 1}{\alpha j \omega T + 1} = k \frac{(1+j\omega T)(1 - j\alpha\omega T)}{1+\alpha^2 T^2 \omega^2}$$

$$= k \frac{[1 + j(\omega T - \alpha \omega T) + \alpha \omega^2 T^2}{1+\alpha^2 T^2 \omega^2}$$

$$= k \left\{ \left[\frac{1+\alpha\omega^2 T^2}{1+\alpha^2 T^2 \omega^2} + \frac{j\omega T(1-\alpha)}{1+\alpha^2 T^2 \omega^2} \right] \right\}$$

$$\Rightarrow \mathrm{Real}(G(j\omega)) = k \frac{1+\alpha\omega^2 T^2}{1+\alpha^2 T^2 \omega^2}$$

$$\mathrm{Img}(G(j\omega)) = k \frac{\omega T(1-\alpha)}{1+\alpha^2 T^2 \omega^2}$$

Let $x \hat{=} k \dfrac{(1+\alpha\omega^2 T^2)}{1+\alpha^2 T^2 \omega^2}$

$y \hat{=} k \dfrac{\omega T(1-\alpha)}{1+\alpha^2 T^2 \omega^2}$

$a = \dfrac{k}{2\alpha}(1+\alpha)$; $r = \dfrac{k(1-\alpha)}{2\alpha}$

Then it can be shown that

$$(x-a)^2 + y^2 = r^2$$

Nyquist of a lead controller

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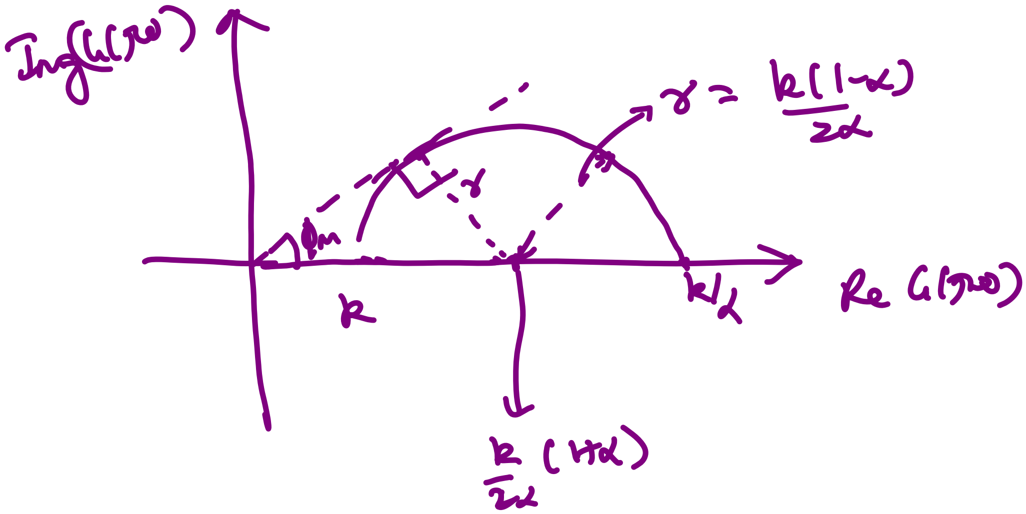
Therefore the Nyquist plot of

$$G(s) = k\,\frac{Ts+1}{\alpha Ts+1}$$ is given by



maximum phase is ϕ_m then

$$\sin\phi_m = \frac{\gamma}{a} = \frac{1-\alpha}{1+\alpha}.$$

Note also that when $\omega = \omega_m = \frac{1}{T\sqrt{\alpha}}$ then

$$\mathrm{Re}\,G(j\omega_m) = k\,\frac{(1+\alpha\omega_m^2T^2)}{1+\alpha^2T^2\omega_m^2} = \frac{k\,2}{1+\alpha}$$

$$\mathrm{Img}\,G(j\omega_m) = k\,\frac{\omega_m T(1-\alpha)}{1+\alpha^2T^2\omega_m^2} = \frac{k}{\sqrt{\alpha}}\,\frac{(1-\alpha)}{1+\alpha}$$

Maximum phase of a lead term

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and

$$\sin \underline{\angle G(j\omega_m)} = \frac{\frac{k}{\sqrt{\alpha}}\frac{(1-\alpha)}{1+\alpha}}{\sqrt{\frac{\cancel{4}k^2}{(1+\alpha)^2} + \frac{k^2}{\alpha}\frac{(1-\alpha)^2}{(1+\alpha)^2}}}$$

$$= \frac{\frac{k}{\sqrt{\alpha}}\frac{(1-\alpha)}{1+\alpha}}{\frac{k}{(1+\alpha)\sqrt{\alpha}}\sqrt{\cancel{4}\alpha + 1+\alpha^2 - 2\alpha}}$$

$$= \frac{1-\alpha}{\sqrt{(1+\alpha)^2}} = \frac{1-\alpha}{1+\alpha}$$

$\therefore$ The frequency at which maximum phase is achieved is $\omega_m = \frac{1}{T\sqrt{\alpha}}$

The <u>magnitude at this</u> frequency ω_m is

$$\sqrt{\frac{(2k)^2}{(1+\alpha)^2} + \frac{k^2(1-\alpha)^2}{\alpha(1+\alpha)^2}} = \frac{k}{(1+\alpha)\sqrt{\alpha}}\sqrt{4\alpha+(1-\alpha)^2}$$

$$= \frac{k}{\sqrt{\alpha}}.$$

Summary

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Summary: The term

$$G(s) = k\,\frac{Ts+1}{\alpha T s+1} \qquad \begin{array}{c} 0<\alpha<1 \\ T>0 \\ k>0 \end{array}$$

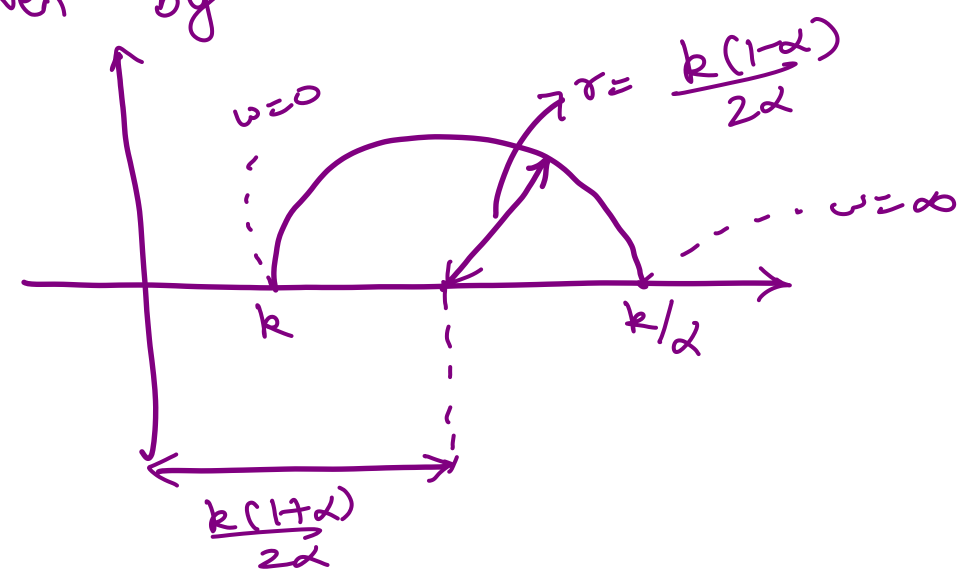
is called a lead term.

- The phase of $G(j\omega) \geq 0$ for $\omega \in [0,\infty)$
- The maximum phase of $G(j\omega)$ is when $\omega = \omega_m = \frac{1}{T\sqrt{\alpha}}$ with $|G(j\omega_m)| = \frac{k}{\sqrt{\alpha}}$ and max phase $\angle G(j\omega_m) =: \phi_m$ satisfies

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

The Nyquist of $G(s) = k\,\frac{Ts+1}{\alpha Ts+1}$

is given by



Lead Controller design

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Lead Compensator Design (steps)

Step 1: Choose k to satisfy static error constants (K_v)

Step 2: Using this k, draw a Bode diagram of $G_1(s) = kG(s)$, and evaluate the phase margin

Step 3: Determine the necessary phase angle needed to meet design specs.

Step 4: Let the extra phase needed be ϕ_{extra}. Then the phase that the controller should provide is given by $\phi_m = \phi_{extra} + (6^o - 10^o)$. Determine α from $\sin \phi_m = \frac{1-\alpha}{1+\alpha}$.

Step 5: Find the frequency ω_c where $|G_1(j\omega_c)| = -20 \log \left(\frac{1}{\sqrt{\alpha}} \right)$. ω_c is the new gain cross over frequency. We design ω_m to be equal to ω_c. Therefore $\frac{1}{T\sqrt{\alpha}} = \omega_c$. Therefore $T = \frac{1}{\omega_c \sqrt{\alpha}}$

Step 6: compensator is given by $G_c(s) = k\frac{Ts+1}{\alpha Ts+1}$

Step 7: Check the design. If it is not satisfactory, one may have to iterate.

Example

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Example

- For a given plant $G(s) = \frac{4}{s(s+2)}$ design a lead controller so that the resulting unity feedback closed loop system has $GM > 10$, $PM > 50^o$ and $K_v = 20$.

Design Steps:

Step 1: Choose k to satisfy static error constants (K_v)

⋆

$$
\begin{aligned}
& K_v = 20 \Rightarrow \lim_{s\to 0} sG_c(s)G(s) = 0 \\
\Rightarrow \quad & \lim_{s\to 0} s\left(k\frac{Ts+1}{\alpha Ts+1}\right)\left(\frac{4}{s(s+2)}\right) = 20 \Rightarrow 2k = 20 \\
\Rightarrow \quad & k = 10
\end{aligned}
$$

Step 2-3

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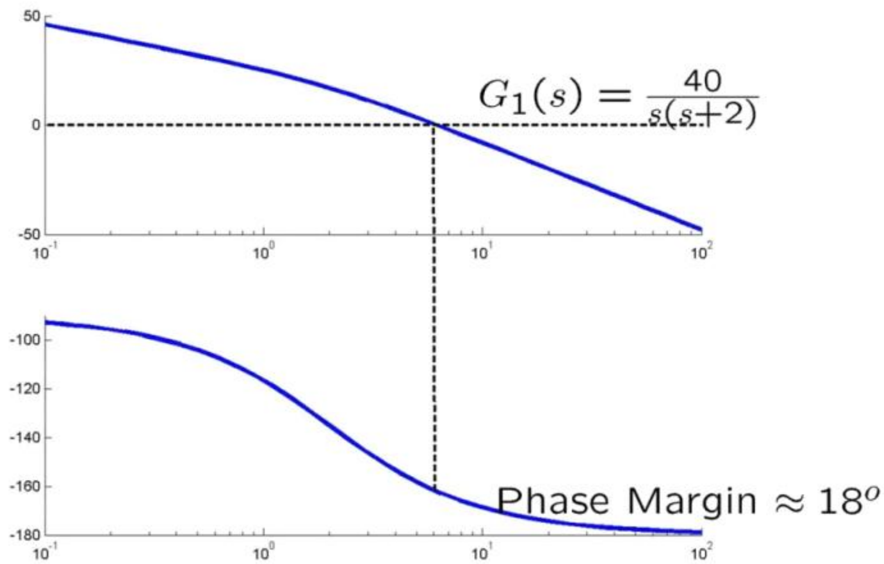
Example

Step 2: Draw Bode diagram of $G_1(s) = kG(s)$ and find PM



Step 3: $\phi_{extra} = 50 - 18 = 32^o$, Therefore $\phi_m \approx 32 + 6 = 38^o$

Steps 4-6

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Example

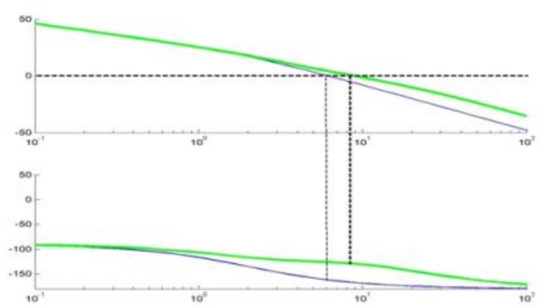
Step 4: Determine α from $\sin\phi_m = \frac{1-\alpha}{1+\alpha}$, i.e.,
$\sin(38^o) = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = 0.24$

Step 5: Find the frequency ω_c where $|G_1(j\omega_c)| = -20\log\left(\frac{1}{\sqrt{\alpha}}\right) = -6.2\ dB$. From bode plot this occurs at $\omega_c = 9\ rad/s$. Then $T = \frac{1}{\omega_c\sqrt{\alpha}} = 0.227$

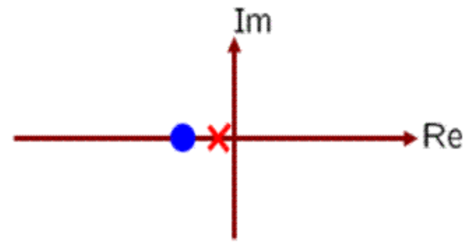
Step 6: Therefore

$$G_c(s) = k\frac{Ts+1}{\alpha Ts+1} = 10\ \frac{0.227s+1}{0.054s+1}$$

Step 7: valid design: $PM = 50.7^o$, $GM = \infty$



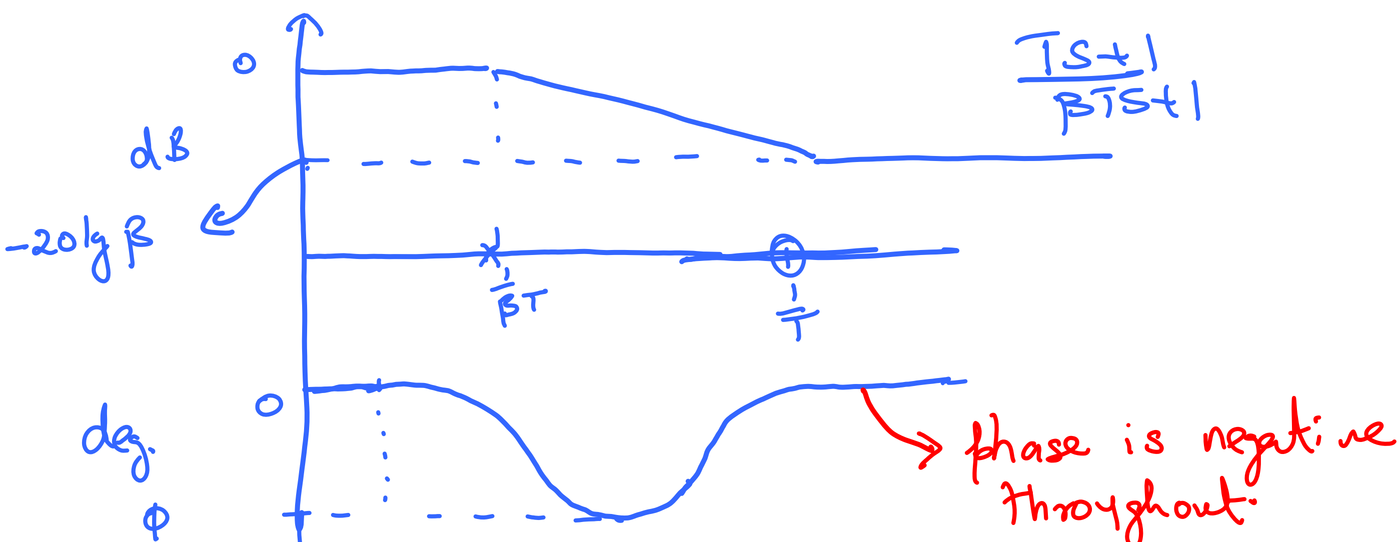
Frequency Domain Design (Lag Compensator)



- General form of a Lag Compensator:

$$G_c(s) = k\frac{Ts+1}{\beta Ts+1} \quad \beta > 1$$

- The main use of the lag compensator is to drag the O.L. magnitude down so as to provide sufficient phase margin
- Compare to the lead, which pushed the O.L. phase plot up to get correct phase margin
- Both share similar structure, but note that different order of poles of zeros. A lead controller acts similar to a PD controller, a lag controller acts similar to a PI controller



Design

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Lag Compensator Design (steps)

Step 1: Choose k to satisfy static error constants (K_v)

Step 2: Using this k, draw a Bode diagram of $G_1(s) = kG(s)$, and determine the required phase margin. Required PM = PM specified $+10^o$. Find the frequency ω_c where $\angle(G_1(j\omega_c))$ is equal to required PM. ω_c is the new gain cross over frequency.

Step 3: Choose the corner frequency of the zero

- ⋆ We want to change the magnitude plot without changing the phase plot at the new crossover frequency
- ⋆ Therefore, choose the zero at $1/T$ to be around 1 decade below the new corner frequency ω_c

Step 4: Determine β and the pole location...

- ⋆ We now examine $|G_1(j\omega_c)|$ to find out how much it is greater than 0 dB. This is equal to $20\log\beta$ i.e.

$$0\ (dB) - |G_1(j\omega_c)|\ (dB) = -20\log\beta$$

$$|G G_c(j\omega_c)| = 1$$

$$\Rightarrow \left| G(j\omega_c)\, k\, \frac{Tj\omega_c + 1}{\beta T j\omega_c + 1} \right| = 1$$

$$\Rightarrow |kG(j\omega_c)| \left| \frac{Tj\omega_c + 1}{\beta T j\omega_c + 1} \right| = 1$$

$$\Rightarrow 20\lg|G_1(j\omega_c)| + 20\lg\left|\frac{jT\omega_c + 1}{j\beta T\omega_c + 1}\right| = 0$$

Analysis

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$$\frac{1}{T} \ll \omega_c$$

$$\Rightarrow \frac{1}{\beta T} \ll \omega_c$$

$$\therefore \frac{|jT\omega_c + 1|}{|j\beta T\omega_c + 1|}$$

$$= \frac{T}{\beta T} \frac{|j\omega_c + 1/T|}{|j\omega_c + 1/\beta T|}$$

$$\simeq \frac{1}{\beta}$$

$$\therefore 20\lg|G_1(j\omega_c)| + 20\lg\frac{1}{\beta} = 0$$

$$\Rightarrow 20\lg\beta = 20\lg|G_1(j\omega_c)|$$

— x —

Example

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Example

- For a given plant $G(s) = \frac{1}{s(s+1)(0.5s+1)}$ design a ~~lead~~ lag controller so that the resulting unity feedback closed loop system has $GM > 10$, $PM > 40^o$ and $K_v = 5$.

Design Steps:

Step 1: Choose k to satisfy static error constants (K_v)

$\star$

$$
\begin{aligned}
& K_v = 5 \Rightarrow \lim_{s \to 0} sG_c(s)G(s) = 0 \\
\Rightarrow \quad & \lim_{s \to 0} s \left(k \frac{Ts+1}{\beta Ts+1} \right) \left(\frac{1}{s(s+1)(0.5s+1)} \right) = 5 \Rightarrow k = 5 \\
\Rightarrow \quad & k = 5
\end{aligned}
$$

Step 2

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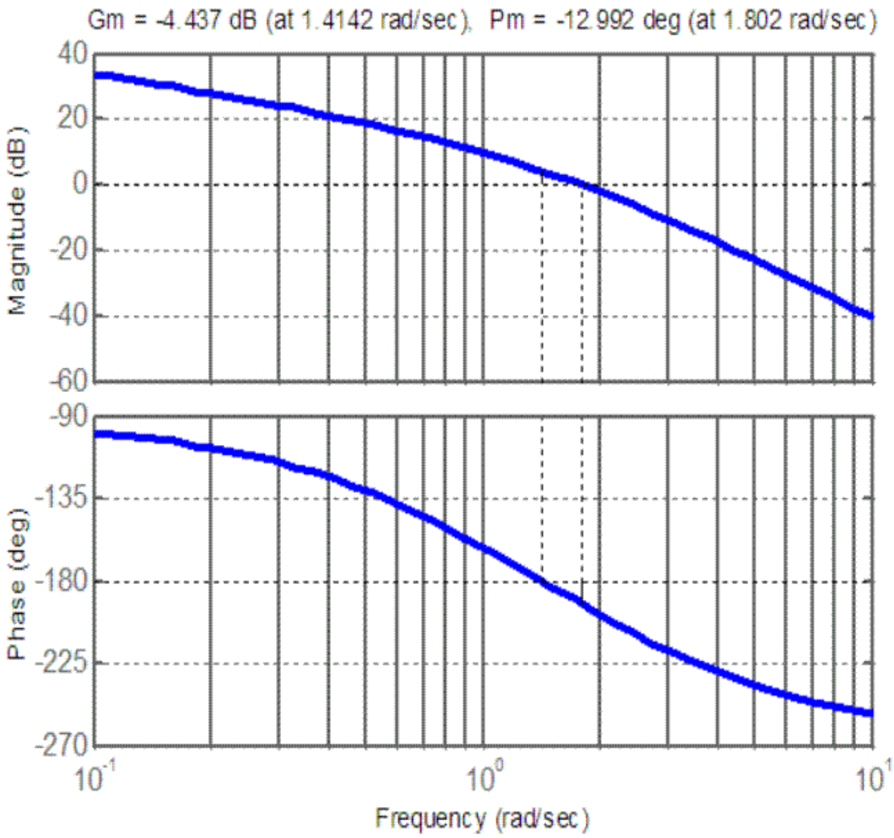
Step 2: Draw Bode diagram of $G_1(s) = kG(s)$



- Required PM $= 40^o + 10^o = 50^o$
- ω_c is that frequency where $\angle(G_1(j\omega_c) = PM - 180 = -130^o$. Therefore $\omega_c = 0.5\ rad/s$ (from the bode plot)

Steps 3 and 4

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Step 3: Choose the corner frequency of the zero

- ⋆ Choose the zero at $1/T$ to be around 1 decade below the new corner frequency ω_c; i.e. $\frac{1}{T} = 0.05$ which implies $T = 20$.

Step 4: Determine β

- ⋆ $|G_1(j\omega)| = 20\ dB$ at $\omega = \omega_c = 0.5\ rad/s$
  Therefore $20 \log \beta = 20 \Rightarrow \beta = 10$

$$G_c(s) = \frac{5(20s + 1)}{200s + 1}$$

Results

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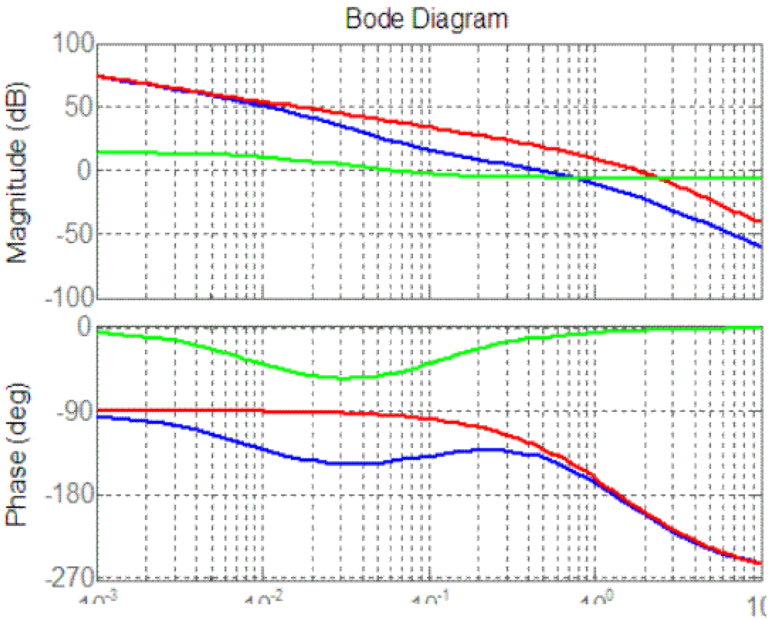

- Frequency Response
- Gain Margin = 14.3dB
- Phase Margin = 42 deg
- Specifications met.
- Green = $G_c(s)$
- Red = $G_1(s)$
- Blue = $G_cG_1(s)$