Building Blocks

Tuesday, October 27, 2009

11:41 PM

- Consider any proper transfer function

$$G(s) = \frac{p(s)}{q(s)} \; ; \quad \deg(p(s)) \le \deg(q(s)).$$

① First convert to the form

$$G(s) = d + \frac{a(s)}{b(s)}$$ where d is a constant and $\deg(a(s)) < \deg b(s)$

② $\tilde{G}(s) = \frac{a(s)}{b(s)}$ where $\deg a(s) < \deg b(s)$

③ <u>Expand using partial fractions</u>:

- $a(s)$ and $b(s)$ are polynomials in s
- therefore if $\alpha + j\beta$ is a root of b then $\alpha - j\beta$ is also a root

$$\therefore \quad \tilde{G}(s) = \frac{a(s)}{b(s)} = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \dots + \frac{A_n}{(s-s_n)}$$

where $\deg(b) = n$. ($\therefore$ n roots, $s_1, s_2 \dots s_n$).

assuming no repeated roots.

Complex Roots

Tuesday, October 27, 2009
11:43 PM

Either s_i are real or if s_i is a complex root then $\bar{s}_i$ is also a root.

- Suppose $\alpha+j\beta$ and $\alpha-j\beta$ are roots then

$$\tilde{G}(s) = \frac{A_1}{s-s_1} + \cdots + \underbrace{\frac{A_i}{s-(\alpha+j\beta)} + \frac{A_{i+1}}{s-(\alpha-j\beta)}}_{\shortparallel} + \cdots + \frac{A_n}{(s-s_n)}.$$

$$= \frac{A_i(s-(\alpha-j\beta)) + A_{i+1}(s-(\alpha+j\beta))}{(s-(\alpha+j\beta))(s-(\alpha-j\beta))}$$

$$= \frac{As+B}{s^2-(\alpha-j\beta)s-(\alpha+j\beta)s + (\alpha-j\beta)(\alpha+j\beta)}$$

$$= \frac{As+B}{s^2-\alpha s+\cancel{j\beta s}-\alpha s-\cancel{j\beta s} + \alpha^2+\beta^2}$$

$$= \frac{As+B}{s^2-2\alpha s+(\alpha^2+\beta^2)}$$

Decomposition of real-rational transfer functions

Tuesday, October 27, 2009
11:45 PM

→ Therefore, assuming no repeated poles any proper transfer function with can be decomposed into first order terms of the form $\frac{1}{s-a_i}$; $a_i \in \mathbb{R}$

or of the form $\frac{As+B}{s^2+\alpha_i s+\beta_i}$; $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$.

→

the partial fraction expansion has terms of the form

$$\frac{1}{s-a} \quad \text{and} \quad \frac{1}{s^2+2\xi\omega_0 s+\omega_0^2}$$

→ Thus, if we know how these transfer functions "behave" we have characterized all rational proper transfer functions with real coefficients.

First order System

Wednesday, October 28, 2009
4:55 PM

(+) Consider a first order system

$$G(s) = \frac{1}{1 + s/p}$$

which has a pole at $-p$;

We will assume that $p > 0$.

The step response of the system is obtainable by the Laplace inverse of

$$Y(s) = G(s)\,\frac{1}{s}$$

$$= \left(\frac{1}{1 + s/p}\right)\left(\frac{1}{s}\right)$$

Let $\frac{1}{(1 + s/p)}\left(\frac{1}{s}\right) = \frac{A}{1 + s/p} + \frac{B}{s}$

$$A = Y(s)\,(1 + s/p)\Big|_{s=-p}$$

$$= \frac{1}{s}\Big|_{s=-p} = -1/p$$

step Response

Wednesday, October 28, 2009
5:28 PM

$$B = sY(s)\Big|_{s=0} = \left(\frac{1}{1+s/p}\right)\Big|_{s=0} = 1$$

$$\therefore\ Y(s) = -\frac{1}{p}\,\frac{1}{1+s/p} + \frac{1}{s}$$

$$\therefore\ Y(s) = -\frac{1}{s+p} + \frac{1}{s}$$

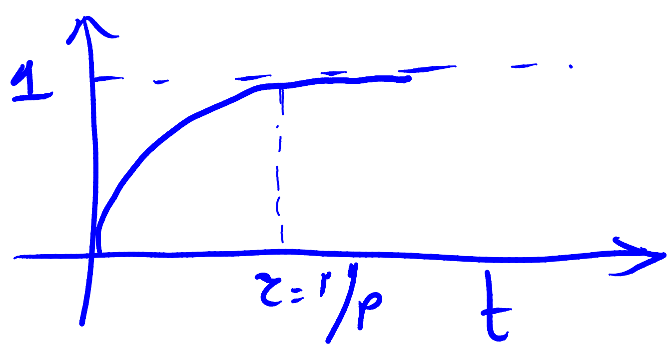
$$y(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s+p}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$= -e^{-pt} + 1$$

$$= \left(1 - e^{-pt}\right)$$

Thus, the step response of

$\frac{1}{1+s/p}$ is given by



Impulse Response

Sunday, November 01, 2009
12:43 AM

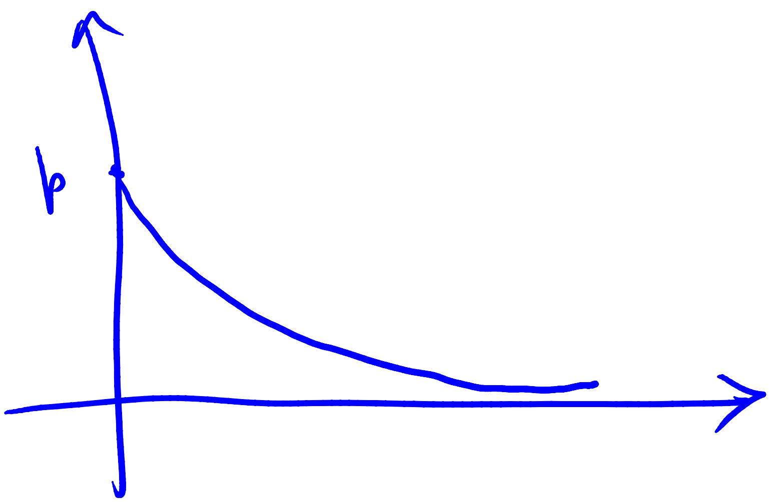
$$G(s) = \frac{1}{1+s/p}$$

Impulse Response is given by

$$g(t) = \mathcal{L}^{-1}\left(\frac{1}{1+s/p}\right)$$

$$= \mathcal{L}^{-1} \frac{1}{\frac{1}{p}(p+s)}$$

$$= p\,\mathcal{L}^{-1} \frac{1}{s+p} = p e^{-pt}$$



Second Order Systems

Tuesday, October 27, 2009
11:53 PM

$$G(s) = \frac{As+B}{s^2+2\xi\omega_0 s+\omega_0^2}$$

$$= \frac{As+B}{s^2+2\xi\omega_0 s+\xi^2\omega_0^2+\omega_0^2-\xi^2\omega_0^2}$$

$$= \frac{As+B}{(s+\xi\omega_0)^2+\omega_0^2(1-\xi^2)}$$

* Roots of the denominator are given by

$$(s+\xi\omega_0)^2+\omega_0^2(1-\xi^2)=0$$

$$\Rightarrow (s+\xi\omega_0)^2 = \omega_0^2(\xi^2-1)$$

$$\Rightarrow s_{1,2} = -\xi\omega_0 \pm \sqrt{\omega_0^2(\xi^2-1)}$$

are the roots.

Location of poles

Tuesday, October 27, 2009
11:54 PM

Case 1: If $\xi > 1$ then roots are real and given by

$$S_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1}$$

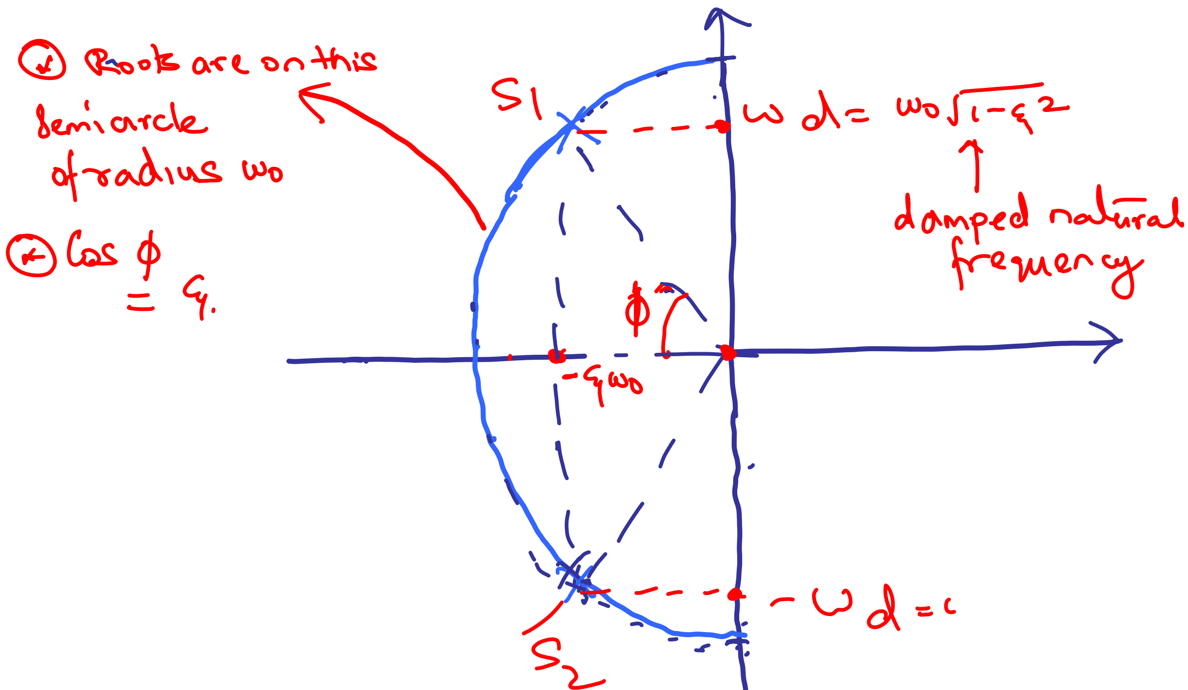
$$S_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$$

Analysis boils down to first order systems.

Case 2: If $\xi < 1$ then roots are complex

$S_1 = -\xi\omega_0 + j\left(\sqrt{1-\xi^2}\right)\omega_0$ and $S_2 = -\xi\omega_0 - j\left(\sqrt{1-\xi^2}\right)\omega_0$

Note that, $|S_1| = |S_2| = \sqrt{\xi^2\omega_0^2 + (1-\xi^2)\omega_0^2} = \omega_0$



Underdamped, critically damped, overdamped

Wednesday, October 28, 2009
6:03 PM

- when $0 < \zeta < 1$ we say system is underdamped
- when $\zeta > 1$ we say system is overdamped.
- when $\zeta = 1$ we say system is critically damped.

Underdamped case

Tuesday, October 27, 2009
11:57 PM

ⓐ: Case $\xi < 1$; imaginary roots.

$$G(s) = \frac{As+B}{s^2+2\xi\omega_0 s+\omega_0^2} = \frac{As+B}{s^2+2\xi\omega_0 s+\omega_0^2}$$

$$= \frac{As+B}{s^2+2\xi\omega_0 s+(\xi\omega_0)^2+\omega_0^2-(\xi\omega_0)^2}$$

$$\therefore\; G(s) = \frac{As+B}{(s+\xi\omega_0)^2+\omega_0^2(1-\xi^2)}$$

Let $\bar{\omega} = \sqrt{\omega_0^2(1-\xi^2)} = \omega_0\sqrt{1-\xi^2}$

(*) Then

$$G(s) = \frac{As+B}{(s+\xi\omega_0)^2+\omega_d^2}$$

$$= \frac{A(s+\xi\omega_0)-A\xi\omega_0+B}{(s+\xi\omega_0)^2+\omega_d^2}$$

$$= [A]\frac{s+\xi\omega_0}{(s+\xi\omega_0)^2+\omega_d^2} + \left[\frac{B-A\xi\omega_0}{\omega_d}\right]\frac{\omega_d}{(s+\xi\omega_0)^2+\omega_d^2}$$

Impulse Response

Wednesday, October 28, 2009
5:59 PM

- The impulse response is given by

$$g(t) = \mathcal{L}^{-1} A \frac{(s+\xi\omega_0)}{(s+\xi\omega_0)^2+\omega_d^2} + \frac{B-A\xi\omega_0}{\omega_d} \mathcal{L}^{-1} \frac{\omega_d}{(s+\xi\omega_0)^2+\omega_d^2}$$

$$= \left[A\, e^{-\xi\omega_0 t} \cos\omega_d t + \frac{B-A\xi\omega_0}{\omega_d} e^{-\xi\omega_0 t} \sin(\omega_d t)\right]$$

$$= e^{-\xi\omega_0 t}\left[A \cos\omega_d t + \frac{B-A\xi\omega_0}{\omega_d} \sin\omega_d t\right]$$

$$= e^{-\xi\omega_0 t}\left[A \cos\omega_d t + A' \sin\omega_d t\right]$$

$$A' = \frac{B-A\xi\omega_0}{\omega_d}$$

Impulse response

Wednesday, October 28, 2009
12:00 AM

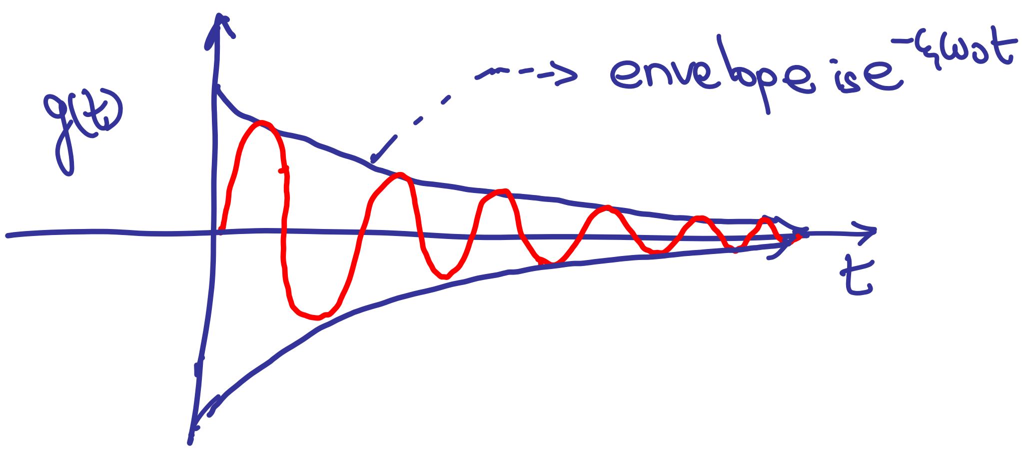
$$= e^{-\xi\omega_0 t}\left[\frac{A}{\sqrt{A^2+A'^2}}\cos\omega_d t + \frac{A'}{\sqrt{A^2+A'^2}}\sin\omega_d t\right]$$

$$= e^{-\xi\omega_0 t}\left[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right]$$

$$= e^{-\xi\omega_0 t}\sin(\omega_d t + \theta);$$

where

$$\sin\theta = \frac{A}{\sqrt{A^2+A'^2}}$$



Step Response

Wednesday, October 28, 2009
12:02 AM

* $$G(s) = \frac{As + B}{s^2 + 2\xi\omega_0 s + \omega_0^2}$$

is the transfer function.

* For the step response, the input is a step $u_s(t)$ to the system G. and the output is

$$Y(s) = G(s)\,U_s(s)$$

$$= G(s)\,\frac{1}{s}$$

$$= \frac{As + B}{(s^2 + 2\xi\omega_0 s + \omega_0^2)\,s}$$

Let

$$Y(s) = \left[\frac{As + B}{(s + \xi\omega_0)^2 + \omega_0^2(1-\xi^2)}\right]\frac{1}{s}$$

$$= \left(\frac{As + B}{(s + \xi\omega_0)^2 + \omega_d^2}\right)\frac{1}{s}$$

Step Response

Wednesday, October 28, 2009
12:03 AM

Let
$$Y(s) = \frac{Cs + D}{(s+\xi\omega_0)^2 + \omega_d^2} + \frac{E}{s}$$

$$E = Y(s)(s)\Big|_{s=0} = \frac{B}{\omega_d^2 + \xi^2\omega_0^2} = \frac{B}{\omega_0^2(1-\xi^2) + \xi^2\omega_0^2}$$
$$= \frac{B}{\omega_0^2}$$

$$\therefore \quad Y(s) = \frac{Cs^2 + Ds + E\left[s^2 + 2\xi\omega_0 s + \omega_0^2\xi^2 + \omega_d^2\right]}{s(s^2 + 2\xi\omega_0 s + \omega_0^2)}$$

$$\Rightarrow \frac{As+B}{s(s^2+2\xi\omega_0 s+\omega_0^2)} = \frac{Cs^2 + Ds + E\left[s^2 + 2\xi\omega_0 s + \omega_0^2\right]}{s(s^2 + 2\xi\omega_0 s + \omega_0^2)}$$

$$\Rightarrow \quad As + B = Cs^2 + Ds + Es^2 + 2\xi\omega_0 s E + E\omega_0^2$$

$$\Rightarrow \quad As + B = (C+E)s^2 + (D + 2\xi\omega_0 E)s + E\omega_0^2$$

$$\Rightarrow \quad C + E = 0 \quad \Rightarrow \quad C = -E = -\frac{B}{\omega_0^2}$$

$$D + 2\xi\omega_0 E = A$$

$$\Rightarrow \quad D = A - 2\xi\omega_0 E = A - 2\xi\omega_0 \frac{B}{\omega_0^2}$$

$$\therefore \quad Y(s) = \frac{-\frac{B}{\omega_0^2}s + \left(A - 2\xi\omega_0\frac{B}{\omega_0^2}\right)}{(s+\xi\omega_0)^2 + \omega_d^2} + \frac{B}{\omega_0^2}\,\frac{1}{s}$$

Step Response

Wednesday, October 28, 2009
12:05 AM

$$= \frac{-\frac{B}{\omega_0^2}(s+\xi\omega_0)+\frac{B}{\omega_0^2}\xi\omega_0 + A - 2\xi\omega_0\frac{B}{\omega_0^2}}{(s+\xi\omega_0)^2+\omega_d^2} + \frac{B}{\omega_0^2}\cdot\frac{1}{s}$$

$$= \frac{-B}{\omega_0^2}\frac{(s+\xi\omega_0)}{(s+\xi\omega_0)^2+\omega_d^2} + \frac{\left(A-\frac{\xi\omega_0 B}{\omega_0^2}\right)}{\omega_d}\frac{\omega_d}{(s+\xi\omega_0)^2+\omega_d^2}$$

$$+ \frac{B}{\omega_0^2}\cdot\frac{1}{s}.$$

$$y(t) = \frac{-B}{\omega_0^2} e^{-\xi\omega_0 t}\cos\omega_d t + \left(A-\frac{\xi\omega_0 B}{\omega_0^2}\right)\cdot\frac{1}{\omega_d} e^{-\xi\omega_0 t}\sin\omega_d t$$

$$+ \frac{B}{\omega_0^2} u_s(t)$$

$$= \frac{B}{\omega_0^2}\left[1 - e^{-\xi\omega_0 t}\cos\omega_d t + \frac{\omega_0^2 A - \xi\omega_0 B}{B\omega_d} e^{-\xi\omega_0 t}\sin\omega_d t\right]$$

In particular if $A=0$

$$G(s) = \frac{B}{s^2+2\xi\omega_0 s+\omega_0^2}$$

Then $y(t) = \frac{B}{\omega_0^2}\left[1 - e^{-\xi\omega_0 t}\cos\omega_d t - \frac{\xi\omega_0}{\omega_d} e^{-\xi\omega_0 t}\sin\omega_d t\right]$

Step response

Wednesday, October 28, 2009
12:07 AM

$$\therefore\ y(t) = \frac{B}{\omega_0^2}\left[1 - e^{-\xi\omega_0 t}\cos\omega_d t - \frac{\xi\omega_0}{\omega_0\sqrt{1-\xi^2}}e^{-\xi\omega_0 t}\sin\omega_d t\right]$$

$$= \frac{B}{\omega_0^2}\left[1 - e^{-\xi\omega_0 t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)\right]$$

$$= \frac{B}{\omega_0^2}\left[1 - e^{-\xi\omega_0 t}\frac{1}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right)\right]$$

Let $\cos\phi = \xi \Rightarrow \sqrt{1-\xi^2} = \sin\phi$

$$\therefore\ y(t) = \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\left(\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t\right)\right]$$

$$= \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)\right]$$

$\therefore$ Step response of $\dfrac{B}{s^2 + 2\xi\omega_0 s + \omega_0^2}$ is

given by

$$\boxed{\frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)\right]}$$

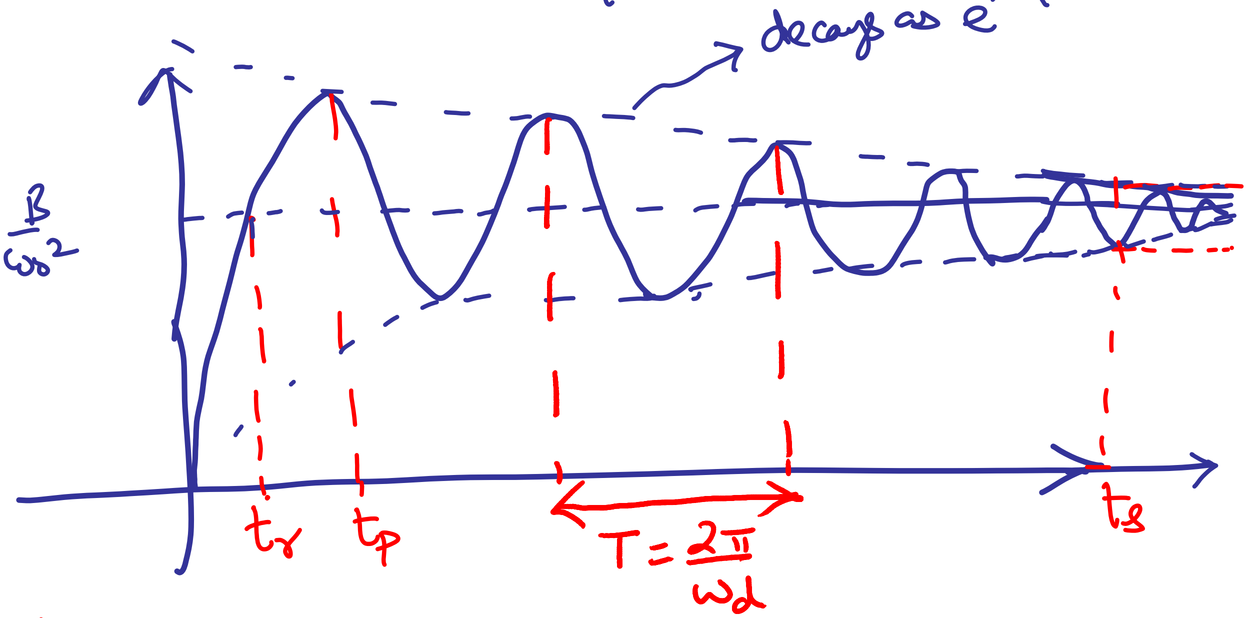
Wednesday, October 28, 2009
12:08 AM

Plots of Step response of *

$$\frac{B}{s^2+2\xi\omega_0 s+\omega_0^2}$$

given by

$$y(t)=\frac{B}{\omega_0^2}\left[1-\frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\phi)\right]$$



* t_r = rise time: the first time the output reaches the steady state value $y_{ss}=B/\omega_0^2$

t_p = peak time; the time the maximum is reached

t_s = settling time: the time after which output remains within 2% of the steady state y_{ss}

y_{ss} = steady state value B/ω_0^2

$$M_p=\frac{y(t_p)-y_{ss}}{y_{ss}}$$

Steady State

Wednesday, October 28, 2009
12:09 AM

Steady State Value:

$$y(t) = \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)\right]$$

$$\omega_d = \omega_0\sqrt{1-\xi^2}$$

$$\cos\phi = \xi.$$

Steady state value obtained by

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)\right]$$

$$= \frac{B}{\omega_0^2} - \frac{B}{\omega_0^2}\lim_{t\to\infty}\left(\frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)\right)$$

$$= \frac{B}{\omega_0^2}.$$

$$\therefore \quad y_{ss} = \frac{B}{\omega_0^2}$$

Rise Time

Wednesday, October 28, 2009
12:10 AM

Rise time t_r

$$y(t) = \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\phi)\right]$$

$$\omega_d = \omega_0\sqrt{1-\xi^2}$$

$$\cos\phi = \xi.$$

$\circledast$ At t_r

$$y(t_r) = y_{ss.} = B/\omega_0^2$$

$$\therefore \quad \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t_r}}{\sqrt{1-\xi^2}}\sin(\omega_d t_r+\phi)\right] = B/\omega_0^2$$

$$\Rightarrow \quad \frac{e^{-\xi\omega_0 t_r}}{\sqrt{1-\xi^2}}\sin(\omega_d t_r+\phi) = 0$$

$$\Rightarrow \quad \sin(\omega_d t_r+\phi) = 0$$

$$\Rightarrow \quad \omega_d t_r + \phi = n\pi$$

$$\Rightarrow \quad t_r = \frac{\pi-\phi}{\omega_d}$$

[$n=1$ as t_r is the first time steady state is reached].

Peak time

Wednesday, October 28, 2009
12:11 AM

Peak time t_p

$$y(t) = \frac{B}{\omega_o^2}\left[1 - \frac{e^{-\xi\omega_o t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \phi)\right]$$

at t_p

$$\frac{dy(t)}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}\left[e^{-\xi\omega_o t}\sin(\omega_d t+\phi)\right] = 0$$

$$\Rightarrow -\xi\omega_o e^{-\xi\omega_o t}\sin(\omega_d t+\phi) + e^{-\xi\omega_o t}\omega_d \cos(\omega_d t+\phi) = 0$$

$$\Rightarrow e^{-\xi\omega_o t}\left[\omega_d \cos(\omega_d t+\phi) - \xi\omega_o \sin(\omega_d t+\phi)\right] = 0$$

$$\Rightarrow e^{-\xi\omega_o t}\,\omega_o\left[\sqrt{1-\xi^2}\cos(\omega_d t+\phi) - \xi\sin(\omega_d t+\phi)\right] = 0$$

$$\Rightarrow \sin\phi\cos(\omega_d t+\phi) - \cos\phi\sin(\omega_d t+\phi) = 0$$

$$\Rightarrow \sin\left[\phi - (\omega_d t+\phi)\right] = 0$$

$$\Rightarrow \sin\omega_d t = 0 \Rightarrow \omega_d t = n\pi$$

$$\Rightarrow \boxed{t_p = \frac{\pi}{\omega_d}}$$

Percentage Overshoot

Saturday, October 31, 2009
11:58 PM

M_p : Peak percentage that captures Overshoot

$$M_p = \frac{y(t_p) - y_{ss}}{y_{ss}} \quad ;$$

where y_{ss} is the steady state value
$t_p = \pi/\omega_d$.

$$y_{ss} = \frac{B}{\omega_0^2}$$

$$y(t_p) = \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \phi)\right]$$

$$= \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t_p}}{\sqrt{1-\xi^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \phi\right)\right]$$

$$= \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t_p}}{\sqrt{1-\xi^2}} \sin(\pi + \phi)\right]$$

$$= \frac{B}{\omega_0^2}\left[1 - \frac{e^{-\xi\omega_0 t_p}}{\sqrt{1-\xi^2}} (-\sin\phi)\right]$$

$$= \frac{B}{\omega_0^2}\left[1 + \frac{e^{-\xi\omega_0 t_p}}{\sqrt{1-\xi^2}} \sin\phi\right]$$

$$= \frac{B}{\omega_0^2}\left[1 + \frac{e^{-\xi\omega_0 t_p}}{\sqrt{1-\xi^2}} \sqrt{1-\xi^2}\right]$$

$$= \frac{B}{\omega_0^2} + \frac{B}{\omega_0^2} e^{-\xi\omega_0 t_p}$$

Percent Overshoot

Sunday, November 01, 2009
12:00 AM

$$\therefore \quad M_p = \frac{\frac{B}{\omega_0^2} e^{-\zeta \omega_0 t_p} - B/\omega_0^2 + B/\omega_0^2}{B/\omega_0^2}$$

$$= e^{-\zeta \omega_0 t_p}$$

$$= e^{-\zeta \omega_0 \frac{\pi}{\omega_d}}$$

$$= \exp\left[-\zeta \omega_0 \frac{\pi}{\omega_0 \sqrt{1-\zeta^2}}\right]$$

$$\therefore \quad \boxed{M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}}}$$

Settling time

Sunday, November 01, 2009
12:02 AM

* For 2% of steady state error for $t \geq t_s$

$$e^{-\zeta \omega_0 t_s} < 0.02$$

$$\Rightarrow \quad t_s \approx \frac{4}{\omega_0 \zeta}$$

* 5% of steady state error for $t \geq t_s$

we have

$$e^{-\zeta \omega_0 t_s} < 0.05$$

$$\Rightarrow \quad t_s \approx \frac{3}{\zeta \omega_0}$$

Summary of Step Response features

Sunday, November 01, 2009
12:04 AM

ⓐ Steady state value $y_{ss} = \frac{B}{\omega_0^2}$

ⓑ Rise time $t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \cos^{-1}(\zeta)}{\omega_0\sqrt{1-\zeta^2}}$

ⓒ Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}}$

ⓓ Overshoot $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$

ⓔ Settling time $t_s = \frac{4}{\zeta\omega_0}$

Step Response Specifications

Sunday, November 01, 2009
12:05 AM

Settling time Specification

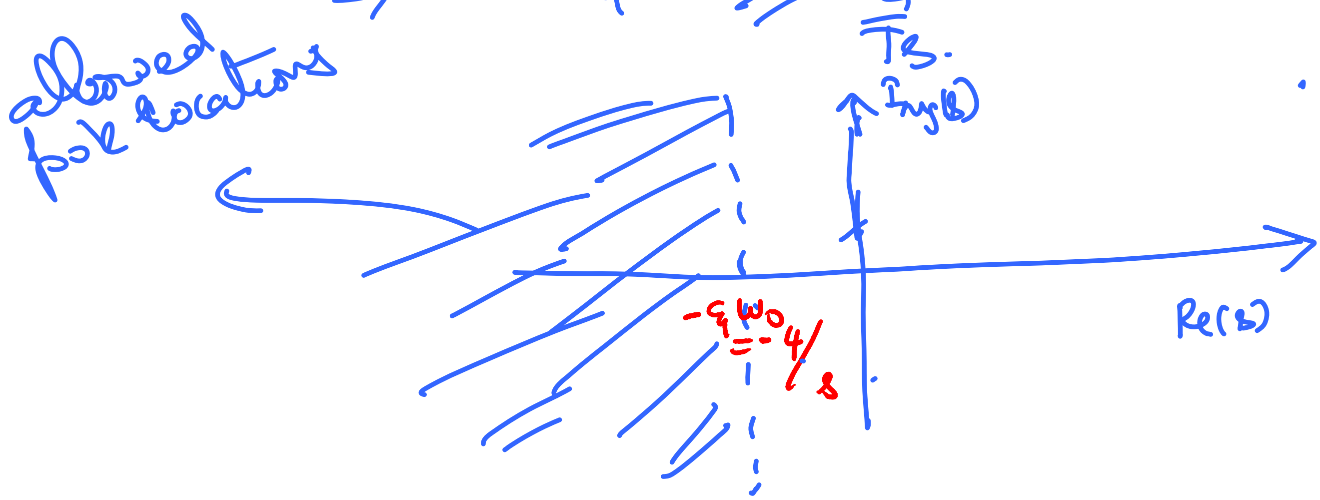
* Typically it is desired that settling time t_s is below a specified value T_s

thus

$$t_s \leq T_s$$

$$\Rightarrow \quad \frac{4}{\zeta \omega_0} \leq T_s$$

$$\Rightarrow \quad \zeta \omega_0 \geq \frac{4}{T_s}.$$



Maximum Overshoot Specification

Sunday, November 01, 2009

12:09 AM

* Typically it is desired that overshoot is small. Thus it is required that $M_p \le M$

$$\Rightarrow \quad e^{-\pi \zeta/\sqrt{1-\zeta^2}} \le M$$

$$\Rightarrow \quad \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \le \ln M$$

$$\Rightarrow \quad \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \le -|\ln M|$$

[Note that $0<M<1 \Rightarrow \ln M < 0 \Rightarrow |\ln M| = -\ln M.$]

$$\Rightarrow \quad \frac{\zeta}{\sqrt{1-\zeta^2}} \geq \frac{|\ln M|}{\pi} =: \alpha$$

$$\Rightarrow \quad \zeta^2 \geq \alpha^2(1-\zeta^2) \;;$$

$$\Rightarrow \quad (1+\alpha^2)\zeta^2 \geq \alpha^2$$

$$\Rightarrow \quad \zeta^2 \geq \frac{\alpha^2}{1+\alpha^2}$$

$$\Rightarrow \quad \zeta > \sqrt{\frac{\alpha^2}{1+\alpha^2}}$$

$$\frac{\cos\phi}{\sin\phi} \geq \alpha$$

$$\Rightarrow \tan\phi \le \frac{1}{\alpha}$$

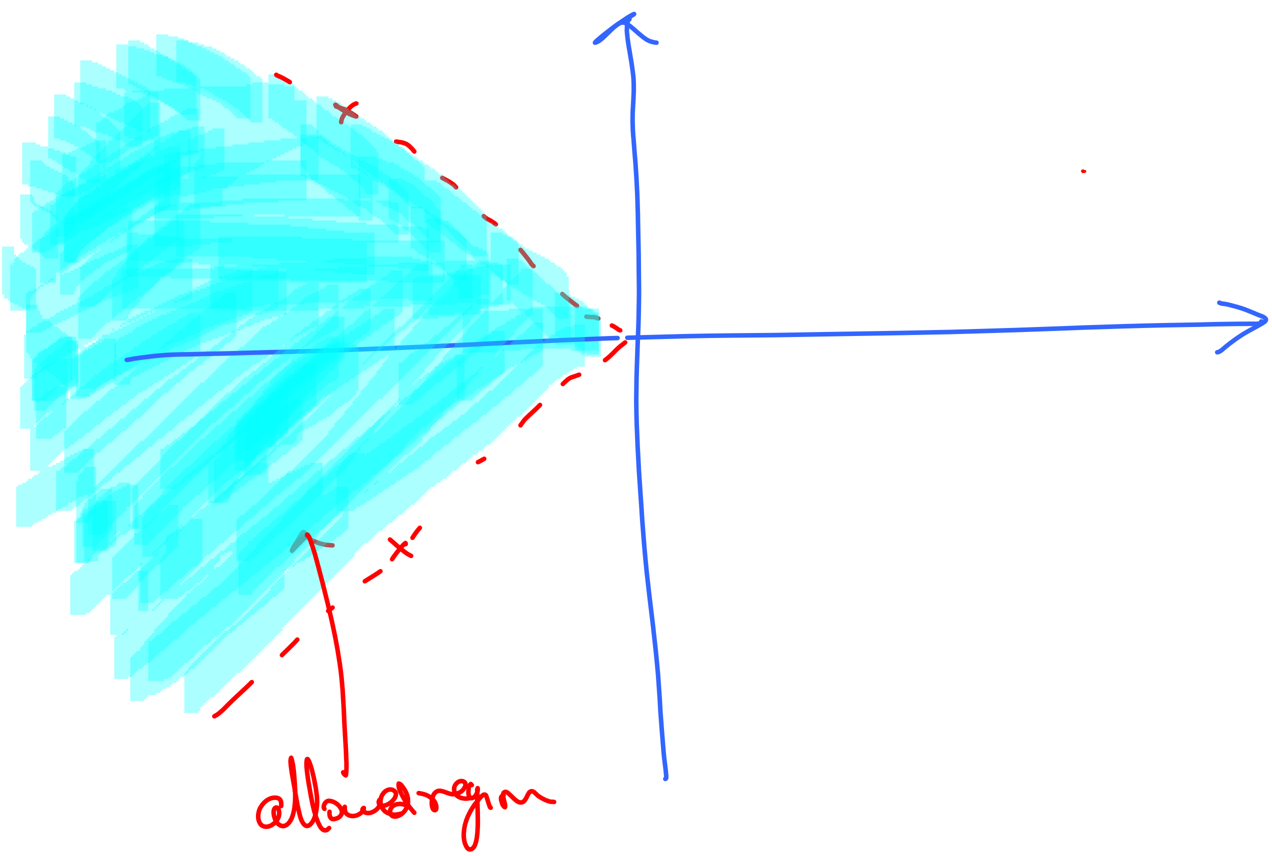
$$\Rightarrow \quad \phi \le \tan^{-1} 1/\alpha$$

Maximum Overshoot Specification

Sunday, November 01, 2009
12:11 AM

Thus damping factor has to be greater than a prespecified value for overshoot to be below a prespecified value.

Pole locations:



Rise time Specification

Sunday, November 01, 2009
12:12 AM

The rise time is given by

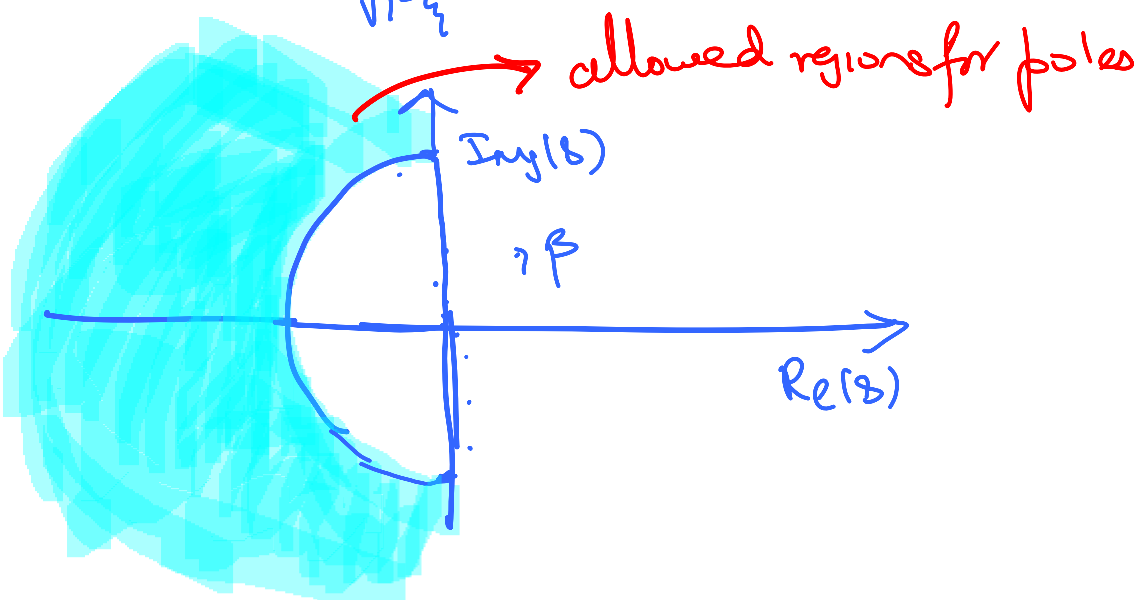
$$t_r = \frac{\pi - \phi}{\omega_d}$$

(*) The rise time should be less than a value T_r. Thus

$$\frac{\pi - \phi}{\omega_d} \le T_r$$

$$\Rightarrow \quad \pi - \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \le \omega_0 \sqrt{1-\zeta^2}$$

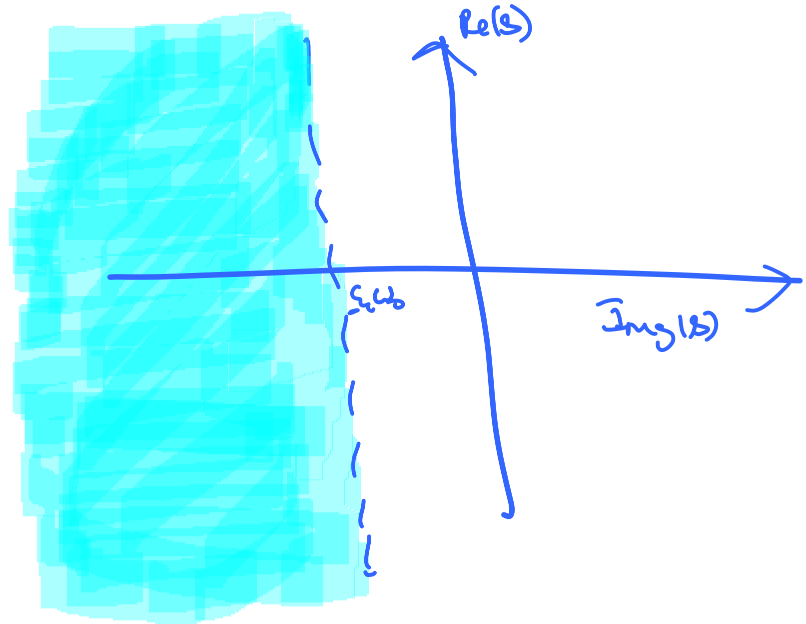
$$\Rightarrow \quad \omega_0 \geq \frac{1}{\sqrt{1-\zeta^2}}\left[\pi - \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] =: \beta$$



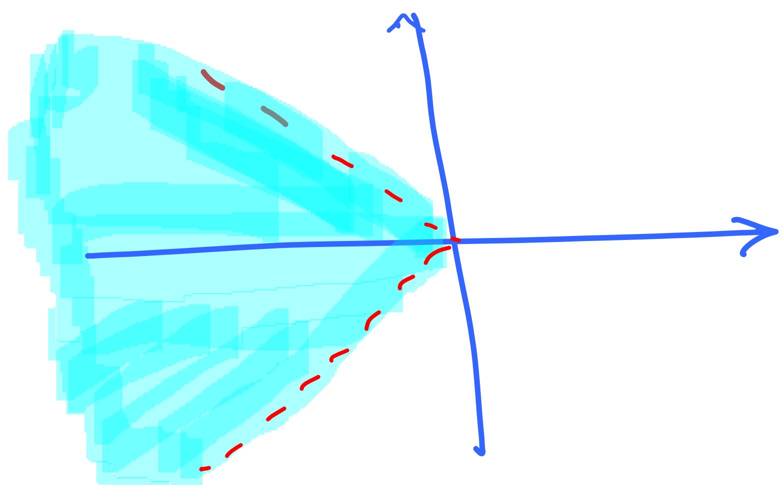
Consolidation of all Specifications

Sunday, November 01, 2009
12:16 AM

ⓐ Settling time $\leq T_s$



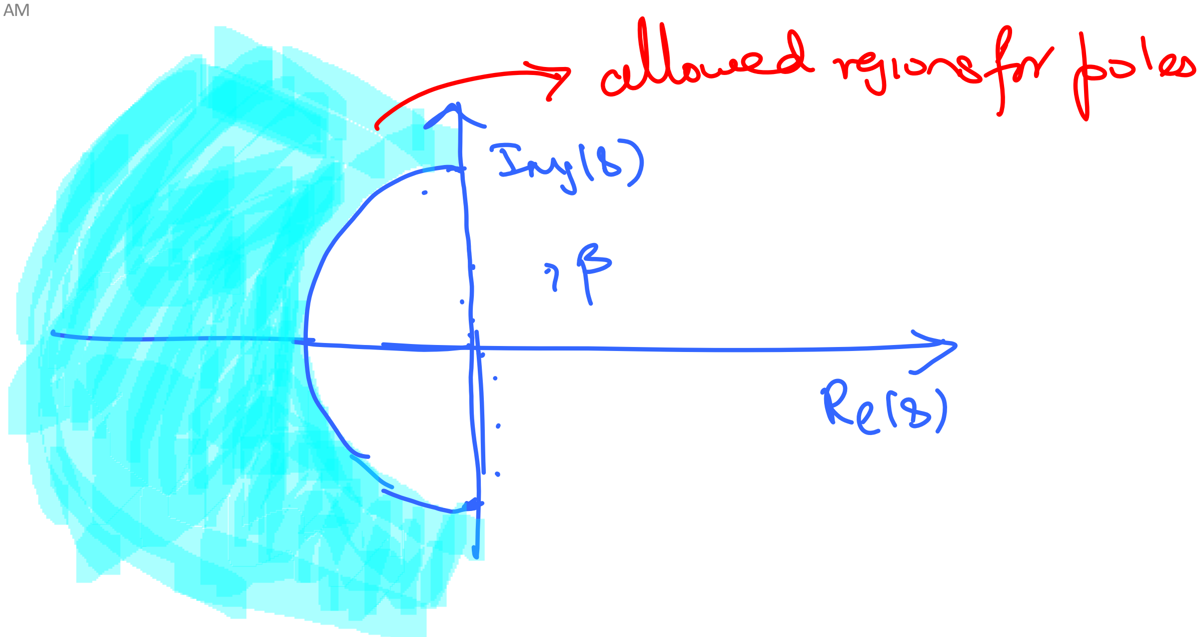
ⓑ Overshoot $M_p \leq M$



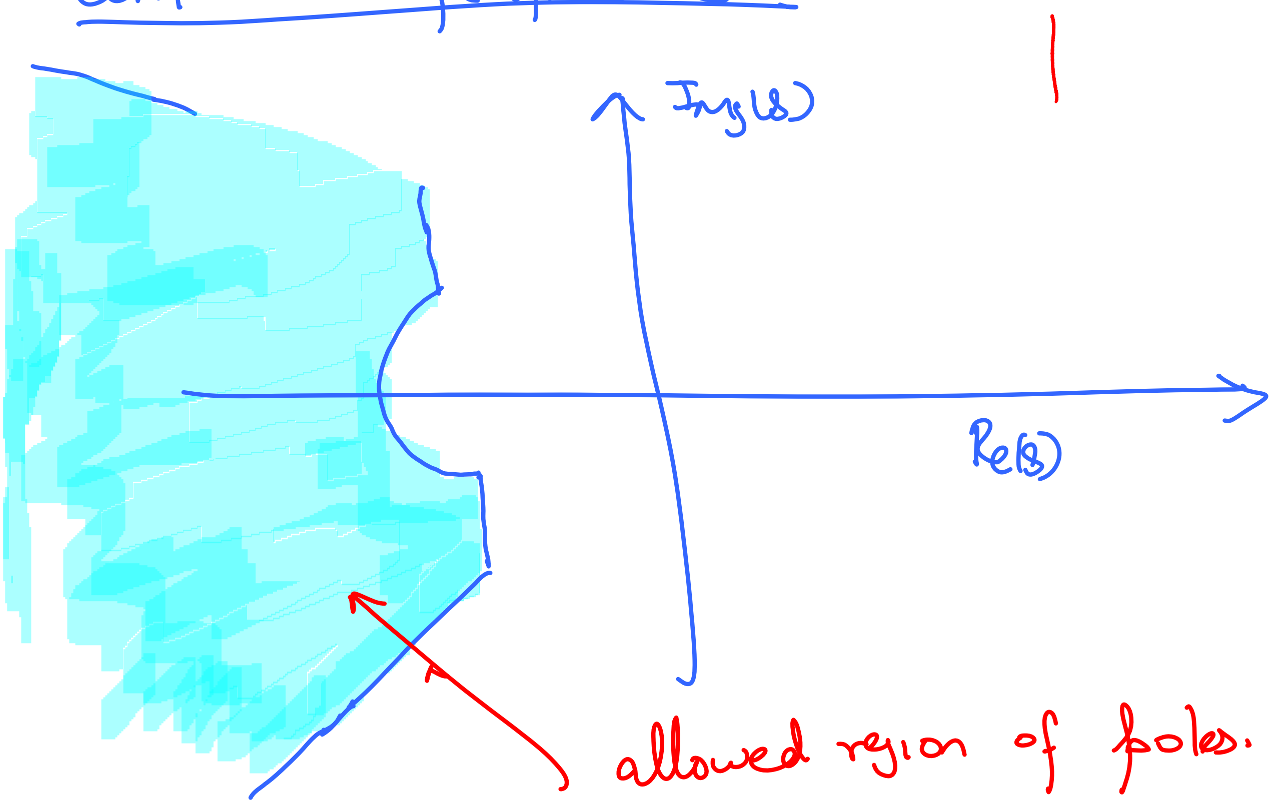
ⓒ Rise time $\leq T_r$

Combined Specifications

Sunday, November 01, 2009
12:21 AM

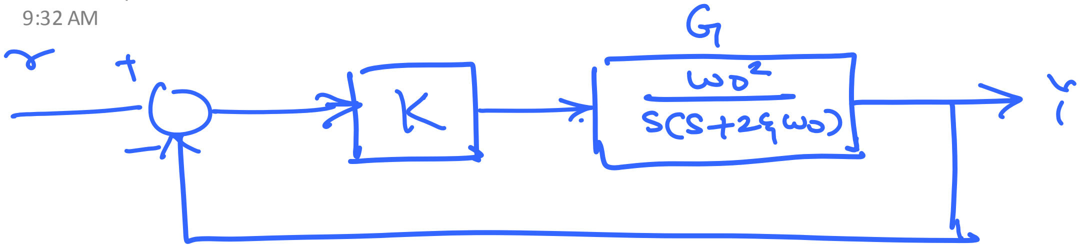


Combined Specifications



Closed-loop prototype

Wednesday, November 04, 2009
9:32 AM



$$\frac{y}{r} = \frac{GK}{1+GK}$$

$$= \frac{\dfrac{\omega_0^2 K}{s(s+2\zeta\omega_0)}}{1 + \dfrac{K\omega_0^2}{s(s+2\zeta\omega_0)}}$$

$$= \frac{\omega_0^2 K}{s^2 + 2\zeta\omega_0 s + K\omega_0^2}.$$

with $K=1$

$$\frac{y}{r} = \frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$$

a standard second order system with

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}};\quad t_r = \frac{\pi - \cos^{-1}\zeta}{\omega_0\sqrt{1-\zeta^2}};\quad t_p = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}};\quad t_s = \frac{4}{\zeta\omega_0}.$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}};\quad BW \approx \omega_0(1-2\zeta^2)$$

$$\omega_r = \omega_0(1-2\zeta^2).$$

Open-loop frequency response

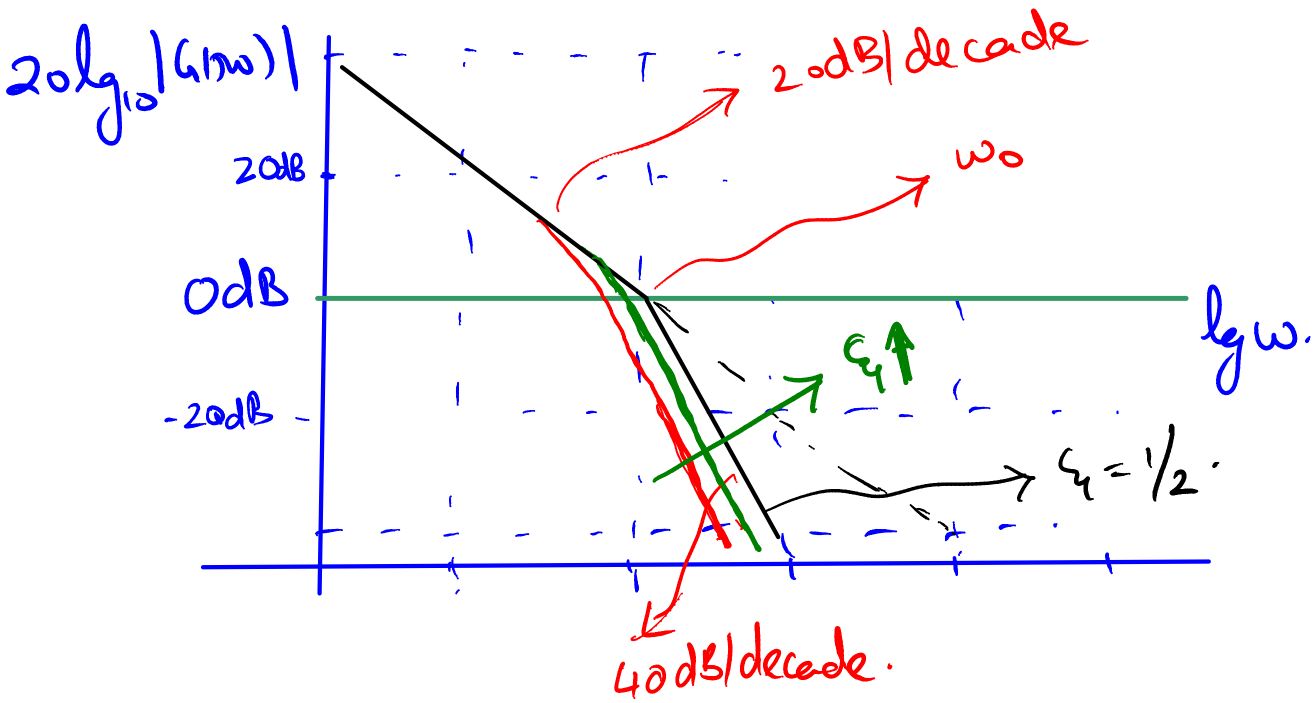
Wednesday, November 04, 2009
9:37 AM

The open-loop part is given by

$$G(s) = \frac{\omega_o^2}{s(s+2\zeta\omega_o)}$$

Lets assume $0 < \zeta < 1$.

The magnitude response is given by



phase Response of the open loop

Wednesday, November 04, 2009
9:51 AM

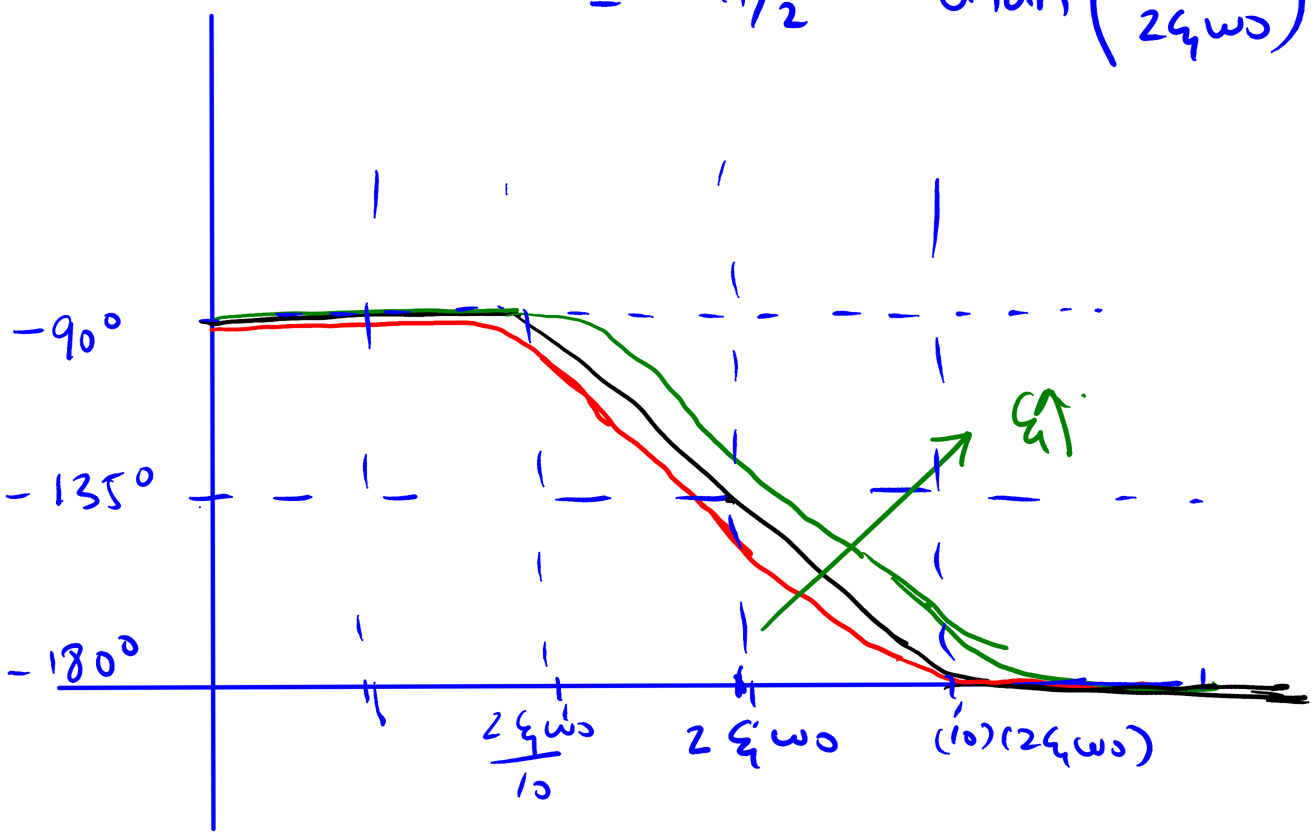
Phase

$$\angle G(j\omega) = \angle \frac{\omega_0^2}{j\omega(j\omega + 2\xi\omega_0)}$$

$$= 0 - \angle j\omega$$

$$- \angle j\omega + 2\xi\omega_0$$

$$= -\pi/2 - \operatorname{atan}\left(\frac{\omega}{2\xi\omega_0}\right)$$



Unity Gain Frequency

Wednesday, November 04, 2009
11:11 AM

Unity gain frequency (UGF) is the frequency at which the magnitude of the transfer function is 1 (which is 0dB).

Thus, if ω_{gc} is the UGF then

$$\left| G(j\omega_{gc}) \right| = 1$$

$$\Rightarrow \left| \frac{\omega_o^2}{-\omega_{gc}^2 + j2\omega_{gc}\omega_o\xi} \right| = 1$$

$$\Rightarrow \omega_o^4 = \omega_{gc}^4 + 4\omega_{gc}^2\omega_o^2\xi^2$$

$$\Rightarrow \omega_{gc}^4 + (4\omega_o^2\xi^2)\omega_{gc}^2 - \omega_o^4 = 0$$

$$\Rightarrow \omega_{gc}^4 + 2(2\omega_o^2\xi^2)\omega_{gc}^2 + 4\omega_o^4\xi^4 - \omega_o^4 = 0$$
$$-4\omega_o^4\xi^4$$

$$\Rightarrow (\omega_{gc}^2 + 2\omega_o^2\xi^2)^2 = \omega_o^4 + 4\omega_o^4\xi^4$$

Thumb rule for UGF

Wednesday, November 04, 2009
11:17 AM

$$\left(\omega_{gc}^2 + 2\omega_0^2\xi^2\right)^2 = \omega_0^4 + 4\omega_0^4\xi^4$$

$$\Rightarrow \omega_{gc}^2 + 2\omega_0^2\xi^2 = \sqrt{\omega_0^4(1+\xi^4 4)}.$$

$$\Rightarrow \omega_{gc}^2 = \omega_0^2\left[\sqrt{1+4\xi^4} - 2\xi^2\right]$$

$$\Rightarrow \omega_{gc} = \omega_0\left[\sqrt{1+4\xi^4} - 2\xi^2\right]^{1/2}.$$

$$\xi = 0 \quad \Rightarrow \quad \omega_{gc} = \omega_0$$

$$\xi = 1 \quad \Rightarrow \quad \omega_{gc} = \omega_0\left(\sqrt{5}-2\right)^{1/2} = (0.24)^{1/2}\omega_0$$

$$\xi = 0.707 \quad \Rightarrow \quad \omega_{gc} = \omega_0\left(\sqrt{2} - 1\right)^{1/2} = (0.41)^{1/2}\omega_0$$

$$\xi = 0.5 \quad \Rightarrow \quad \omega_{gc} = (0.62)^{1/2}\omega_0$$

A good thumb rule is

$$\boxed{\omega_{gc} = \frac{\omega_0}{1.4}}$$

Closed-loop Bandwidth

Wednesday, November 04, 2009
11:27 AM

Closed-loop transfer function is

$$G_{cl} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$\therefore$ Let BW be the frequency such that $\left|G_{cl}(\omega)\right|^2_{\omega=BW} = 1/2$

$$\Rightarrow \left|\frac{\omega_0^2}{-\omega^2 + (2\zeta\omega\omega_0)j + \omega_0^2}\right|^2 = 1/2$$

$$\Rightarrow \left.(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_0^2\right|_{\omega=BW} = 2\omega_0^4$$

$$\Rightarrow (\omega_0^2 - BW^2)^2 + 4\zeta^2 BW^2 \omega_0^2 = 2\omega_0^4$$

$$\Rightarrow BW = \omega_0 \sqrt{1 - 2\zeta^2 + \sqrt{2 + 4\zeta^4 - 4\zeta^2}}$$

$\zeta = 0 \Rightarrow BW = 1.55\,\omega_0 \approx 1.55\,\omega_{gc}$

$\zeta = 0.5 \Rightarrow BW = 1.27\,\omega_0 \approx 1.61\,\omega_{gc}$

$\zeta = 0.7 \Rightarrow BW = 1.01\,\omega_0 \approx 1.577\,\omega_{gc}$

$$\boxed{BW = (1.2 - 1.6)\,\omega_{gc}}$$

Phase Margin

Wednesday, November 04, 2009
11:34 AM

$$\text{Phase margin} = \pi + \angle G(j\omega_{gc})$$

$$= \pi - \frac{\pi}{2} - \operatorname{atan}\left(\frac{\omega_{gc}}{2\zeta\omega_0}\right)$$

$$= \frac{\pi}{2} - \operatorname{atan}\left[\frac{\left(\sqrt{1+4\zeta^4} - 2\zeta^2\right)^{1/2}}{2\zeta}\right]$$

$\zeta = 0.5$; $PM = 51^\circ$

$\zeta = 0.6$; $PM = 59^\circ$

$\zeta = 0.7$; $PM = 65^\circ$

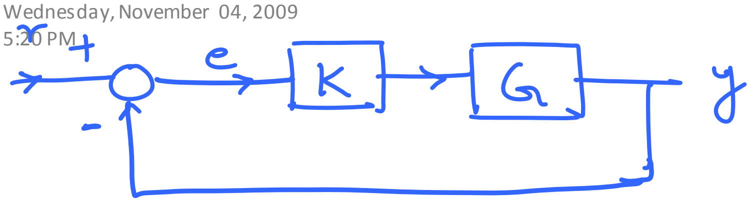
$\zeta = 1$ $PM = 76^\circ$

For most typical systems

$PM = 100\zeta$ in degrees.

Internal model principle

Wednesday, November 04, 2009
5:20 PM



Suppose

$$r = \frac{n_r}{d_r}$$

$$K = \frac{n_K}{d_K}$$

$$G = \frac{n_G}{d_G}$$

$$y = \frac{n_y}{d_y}$$

where (n_r, d_r), (n_K, d_K), (n_G, d_G), (n_y, d_y) are coprime polynomials

Furthermore assume that feedback interconnection is stable i.e.

$n_G n_K + d_G d_K$ has all poles in the rhp.

Note that

$$\frac{e(s)}{r(s)} = \frac{1}{1+GK} = \frac{d_G(s)\, d_K(s)}{n_G(s)\, n_K(s) + d_G(s)\, d_K(s)}$$

Conditions

Wednesday, November 04, 2009
5:22 PM

$$\Rightarrow \quad e(s) = \left[\frac{d_u(s)\, d_k(s)}{n_u(s)\, n_k(s) + d_u(s)\, d_k(s)} \right] r(s).$$

$$= \left[\frac{d_u(s)\, d_k(s)}{n_u(s)\, n_k(s) + d_u(s)\, d_k(s)} \right] \frac{n_r(s)}{d_r(s)}.$$

$$\Rightarrow \quad s\, e(s) = s \left[\frac{d_u(s)\, d_k(s)\, n_r(s)}{n_u(s)\, n_k(s) + d_u(s)\, d_k(s)} \right] \frac{1}{d_r(s)}.$$

Then $\lim_{t \to \infty} e(t) = 0$ if

(a) $s e(s)$ has no poles in the RHP and

(b) $\lim_{s \to 0} s\, e(s) = 0.$

The above follows from the Final value theorem.

Conditions

Wednesday, November 04, 2009
5:27 PM

② From the stability of the closed-loop system we know that $n_u n_k + d_u d_k$ has no rhp roots. Thus

(i) $se(s)$ will have no rhp poles if $d_r(s)$ has no rhp roots or

(ii) Any rhp roots of $d_r(s)$ are cancelled by rhp roots of $d_u(s) d_k(s) n_r(s)$. $n_r(s)$ and $d_r(s)$ are coprime and therefore have no common factor

Therefore any rhp roots of $d_r(s)$ have to be cancelled by roots of $d_u(s) d_k(s)$.

If either (i) or (ii) is satisfied then

$$e(s) = \frac{r}{1+GK}$$

will have no rhp poles (note that $s=0$ is in the rhp) and

$$\lim_{s \to 0} (s\,e(s)) = 0.$$

The important conclusion reached is that

$\lim_{t \to \infty} e(t) = 0$ if all the following conditions are met

(a) The closed-loop interconnection is stable
$[(n_u n_k + d_u d_k)$ has no rhp roots$]$

(b) (i) $r(s) = \frac{n_r(s)}{d_r(s)}$ is such that $d_r(s)$ has no rhp roots

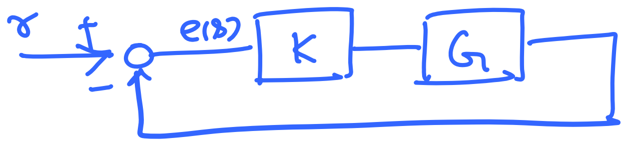
or

(ii) If $r(s) = \frac{n_r}{d_r}$ is such that $d_r(s)$ has rhp roots then the unstable roots (the rhp roots) are present in $d_u d_k$ roots. Thus, a "model" of the reference is captured by the open loop part $\left(G K = \frac{n_u n_k}{d_u d_k}\right)$

Type I systems

Wednesday, November 04, 2009
5:38 PM

Steady State Response to a step



Type 1 System have zero steady state error when the input r is a step

Thus with $r(t) = u_s(t)$

$$e_{ss} \hat{=} \lim_{t \to \infty} e(t) = 0$$

Note that

$$e_{ss} = \lim_{s \to 0} s\, e(s) = \lim_{s \to 0} s \frac{1}{(1+GK)} r(s)$$

$$= \lim_{s \to 0} \left(s \frac{1}{(1+GK)} \cdot \frac{1}{s} \right)$$

$$= \lim_{s \to 0} \left[\frac{1}{1+GK} \right]$$

$$= \frac{1}{1+G(0)K(0)}$$

The "open-loop" transfer function is

$$L(s) = G(s)\,K(s).$$

and we define $K_p \hat{=} \lim_{s \to 0} G(s)\,K(s)$

Conditions

Wednesday, November 04, 2009
5:40 PM

Therefore

$$e_{ss} = \frac{1}{1+K_p}$$

Clearly $e_{ss} = 0$ if $K_p = \infty$

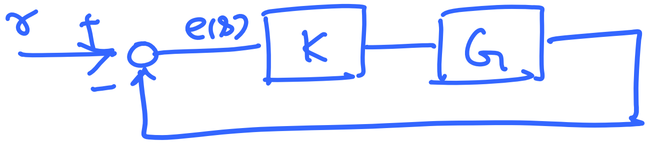
that is if

$$G(0)K(0) = \infty$$

that is if $G(s)K(s)$ has a pole at $s=0$.

Type II Systems

Wednesday, November 04, 2009
5:41 PM



when $r(t) = t\,u_s(t)$

$\therefore \quad e(s) = \frac{1}{1+GK} \cdot r(s) = \frac{1}{1+GK} \cdot \frac{1}{s^2}$

$\therefore \quad r(s) = \frac{1}{s^2} \cdot$

$$\therefore \quad e_{ss} = \lim_{s \to 0} s\,e(s) = \lim_{s \to 0} \left[s \frac{1}{(1+GK)} \frac{1}{s^2} \right]$$

$$= \lim_{s \to 0} \frac{1}{s(1+GK)}$$

$$= \lim_{s \to 0} \frac{1}{s + sGK} = \lim_{s \to 0} \frac{1}{sGK}$$

Let

$$K_v \triangleq \lim_{s \to 0} sGK$$

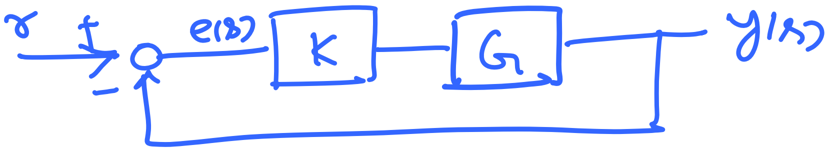
$$\therefore \quad e_{ss} = \lim_{s \to 0} \frac{1}{sGK} = \frac{1}{K_v}$$

Clearly e_{ss} to ramp $= 0$ if $K_v = \infty$ i.e. if $\lim_{s \to 0} (sGK) = \infty$. i.e. if GK has a factor $\frac{1}{s^2}$

Type II Systems have zero steady state error to ramp inputs.

Type III Systems

Wednesday, November 04, 2009
5:44 PM



when $r(t) = t^2 u_s(t)$

$$r(s) = \frac{1}{s^3}$$

$$\Rightarrow \quad e(s) = \frac{r(s)}{1+GK}$$

$$= \frac{1}{s^3}\left(\frac{1}{1+GK}\right)$$

and

$$e_{ss} = \lim_{s\to 0} \frac{s}{s^3} \cdot \frac{1}{(1+GK)}$$

$$= \lim_{s\to 0} \frac{1}{s^2 + s^2 GK}$$

$$= \frac{1}{\lim_{s\to 0}(s^2 GK)} = \frac{1}{K_a}.$$

when e_{ss} due to a parabolic input is zero we says the system is Type III.