Random Processes

* Suppose $X(t)$ is a random process

Then let the mean function be defined by

$$\mu_X(t) = E[X(t)]$$

* Define the Correlation function as

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)^*].$$

* Define the Covariance function as

$$K_X(t_1, t_2) \stackrel{\Delta}{=} E\left[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))^*\right]$$

One can easily show that

$$K_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Markov Random Process

Definition:

(a) A continuous-valued Markov process $X(t)$ satisfies

$$f_X(x_n, t_n \mid x_{n-1}, x_{n-2}, \ldots x_1; t_{n-1}, \ldots t_1)$$
$$= f_X(x_n, t_n \mid x_{n-1}, t_{n-1})$$

for all $x_1, x_2, \ldots x_n$ and for all $t_1 < t_2 < \ldots < t_n$ and for all integers $n > 0$

(b) A discrete-valued Markov process satisfies

$$P_X(x_n, t_n \mid x_{n-1}, \ldots x_1; t_{n-1}, \ldots t_1)$$

$$= P_X(x_n; t_n \mid x_{n-1}; t_{n-1})$$

for all $x_1, \ldots x_n$ and $t_1 < t_2 \ldots < t_n$ for all integers $n > 0$.

Chapman - Kolmogorov Equations

Consider three times $t_3 > t_2 > t_1$ and Markov process random variables at these times $X(t_3)$; $X(t_2)$; $X(t_1)$.

Then

$$f_X(x_3, x_1; t_3, t_1) = \int_{-\infty}^{\infty} f_X(x_3, x_2, x_1; t_3, t_2, t_1)\, dx_2$$

$$= \int_{-\infty}^{\infty} f_X(x_3 \mid x_2, x_1; t_3, t_2, t_1)\, f_X(x_2, x_1; t_2, t_1)\, dx_2$$

$$\Rightarrow f_X(x_3 \mid x_1; t_3; t_1)\, \cancel{f_X(x_1; t_1)} = \int_{-\infty}^{\infty} f_X(x_3 \mid x_2, x_1; t_3, t_2, t_1)\, f_X(x_2 \mid x_1; t_2, t_1)\, \cancel{f_X(x_1; t_1)}\, dx_2$$

$$\Rightarrow f_X(x_3 \mid x_1; t_3; t_1) = \int_{-\infty}^{\infty} f_X(x_3 \mid x_2, x_1; t_3, t_2, t_1)\, f_X(x_2 \mid x_1; t_2, t_1)\, dx_2$$

$$\Rightarrow \boxed{f_X(x_3 \mid x_1) = \int_{-\infty}^{\infty} f_X(x_3 \mid x_2)\, f(x_2 \mid x_1)\, dx_2}$$

where we have dropped t_i for convenience.

Linear Systems With random Inputs

Theorem: Let the random process $X(t)$ be the input to a linear system L with output process $Y(t)$. Then the mean function of the output is given as

$$E[Y(t)] = L\{E(X(t))\}$$

$$= L[\mu_X(t)]$$

Proof: By definition

$$Y(t,\xi) = L[X(t,\xi)]$$

$$\Rightarrow E[Y(t,\xi)] = E[L X(t,\xi)]$$

$$= L[E(X(t,\xi))]$$

The last step (somewhat heuristically) follows as the action of L can be captured by a convolution kernel described the output

$$\int_{-\infty}^{\infty} h(t,\tau)\, u(\tau)\, d\tau$$

for a deterministic input u

Mean

From which it follows that

$$E\left[L\{X(t,\zeta)\}\right] = E\left[\int_{-\infty}^{\infty} h(t,\tau)\, X(\tau,\zeta)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(t,\tau)\, E[X(\tau,\zeta)]\, d\tau$$

$$= \int_{-\infty}^{\infty} h(t,\tau)\, \mu_X(\tau)\, d\tau$$

$$= L[\mu_X(t)].$$

$\square$

Classification

Definitions

Let X and Y be random processes. They are

(a) Uncorrelated if $R_{XY}(t_1, t_2) = \mu_X(t_1)\mu_Y(t_2)$
$\forall\, t_1, t_2$

(b) orthogonal if $R_{XY}(t_1, t_2) = 0.\ \forall\, t_1, t_2$

(c) Independent if for all positive integers n, the nth order pdf of X and Y factors

$$P_{XY}(x_1, y_1, x_2 y_2 \ldots x_n y_n; t_1 t_2, \ldots t_n)$$

$$= P_X(x_1, x_2 \ldots x_n; t_1, t_2 \ldots t_n)\, P_Y(y_1, \ldots y_n, t_1, t_2, \ldots t_n).$$

Stationary Process

Definition

A random process $X(t)$ is stationary if it has the same n^{th} order distribution as $X(t+T)$ for all T and for all positive n.

Note that for stationary processes

the pdf

$$p(x_1, x_2;\ t_1, t_2) = p(x_1, x_2;\ t_1+T, t_2+T)$$

and in particular by choosing $T = -t_2$

$$p(x_1, x_2;\ t_1, t_2) = p(x_1, x_2;\ t_1 - t_2, 0).$$

which implies

$$R_X(t_1, t_2) = E[\,X(t_1)\,(X^*(t_2))\,]$$

$$= E[\,X(t_1 - t_2)\,X^*(0)\,]$$

$$= R_X(t_1 - t_2, 0)$$

and we can define

$$R_X(\tau) \doteq R_X(\tau, 0) = E[X(t+\tau)\,X^*(t)]$$

$\forall\, t.$

Wide Sense stationarity

Definition:

A process $x(t)$ is wide sense stationary if

$$E[x(t+\tau)x(t)] = R(\tau); \quad -\infty < \tau < \infty.$$

independent of t

and

$$E[x(t)] = \mu_x \quad \text{a constant}$$

independent of t.

Wide Sense Stationary and Linear Systems

Note that ~~so~~ if $Y(t)$ is the output random process with $X(t)$ a WSS process as the input to a Linear time-invariant system then

$$E[Y(t)] = L[\mu_X(t)]$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \mu_X(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \mu_X\, d\tau$$

$$= \mu_X \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

$$= \mu_X \left(\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} d\tau \right)\Bigg|_{\omega=0}$$

$$= \mu_X \, H(0)$$

where $H(0)$ is the frequency response representation of the linear system L.

Wide Sense Stationary and Linear Systems

Note that

$$R_{XY}(t_1,t_2) = E[X(t_1)\,Y^*(t_2)]$$

$$= E\left[X(t_1)\int_{-\infty}^{\infty} h^*(z)\,X^*(t_2-z)\,dz\right]$$

$$= \int_{-\infty}^{\infty} h^*(z)\,E[X(t_1)\,X^*(t_2-z)]\,dz$$

$$= \int_{-\infty}^{\infty} h^*(z)\,R_{XX}(t_1,t_2-z)\,dz$$

Similarly

$$R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty} h(z_1)\,R_{XY}(t_1-z_1,t_2)\,dz_1$$

Mean Square Calculus

Sample function Continuity:

$$\lim_{\varepsilon \to 0} x(t+\varepsilon, \xi) = x(t) \quad \forall \xi,$$

Almost Sure Continuity

$$P\left[\lim_{s \to t} x(s) \neq x(t)\right] = 0$$

p-Continuity:

$$\lim_{s \to 0} P\left[\,|x(s) - x(t)| > \varepsilon\right] = 0$$

for each $\varepsilon > 0$.

Continuity in the Mean Square Sense.

Definition:

A random process $X(t)$ is continuous in the mean square sense at the point t if

$$E\left[\,|X(t+\varepsilon)-X(t)|^2\right] \to 0$$

as $\varepsilon \to 0$.

If the above holds for all t then

$X(t)$ is mean square continuous

Mean Square Continuous

Theorem: The random process $X(t)$ is m.s. continuous if and only if

$R_X(t_1, t_2)$ is continuous at the point (t, t) for all t.

Proof:

$$E\left[\,|X(t+\varepsilon) - X(t)|^2\,\right]$$

$$= E[X(t+\varepsilon)X^*(t+\varepsilon)] + E[X(t)X^*(t)]$$

$$- E[X(t+\varepsilon)X^*(t)]$$

$$- E[X^*(t)X(t+\varepsilon)]$$

$$= R_X(t+\varepsilon, t+\varepsilon) + R_X(t)$$

$$- R_X(t+\varepsilon, t) - R_X(t, t+\varepsilon)$$

$\therefore$ The theorem follows.

Mean Square Continuity

<u>Corollary</u>: A wide sense r. process is m.-s continuous for all t if and only if $R_X(\tau)$ is continuous at $\tau = 0$.

Mean Square differentiability

The random process $X(t)$ has a mean square derivative at t if the limit

$$\frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

has a mean square limit.

i.e $\exists$ I such that

$$E\left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - I\right)^2\right] \longrightarrow 0$$

as $\varepsilon \to 0$.

Theorem: A random process $X(t)$ with autocorrelation $R_X(t_1, t_2)$ has a m.s. derivative at time t if and only if

$$\frac{\partial^2 R_X(t_1, t_2)}{\partial t_1 \partial t_2} \text{ exists at } t_1 = t_2 = t.$$

Mean Square derivative

<u>Theorem</u>: If a random process $X(t)$ with mean function $\mu_X(t)$ and correlation function $R_X(t_1, t_2)$ has a m.-s derivative $X'(t)$ then the mean and correlation function of $X'(t)$ are given by

$$\mu_{X'}(t) = \frac{d\mu_X(t)}{dt}$$

$$R_{X'}(t_1, t_2) = \frac{\partial^2 R_X(t_1, t_2)}{\partial t_1 \partial t_2}$$

(*) Note that we have seen earlier that

If $X(t)$ is a Wiener process then

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \min(t_1,t_2)\cdot\alpha$$

and $E(X(t)) = 0 \quad \forall t$

(this was a HW problem).

Clearly

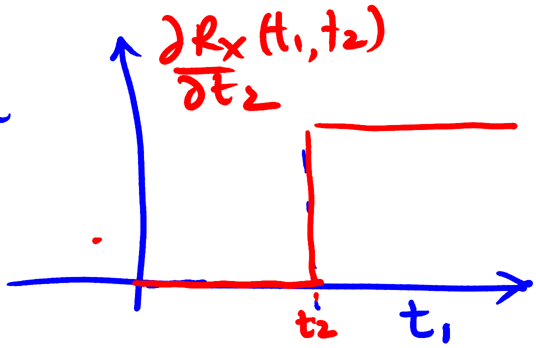
$$\frac{\partial R_X(t_1,t_2)}{\partial t_2} = \frac{\partial}{\partial t_2}(\alpha t_2) \quad \text{if } t_2 < t_1$$

$$= \frac{\partial}{\partial t_2}(\alpha t_1) \quad \text{if } t_1 < t_2$$

$$\therefore \frac{\partial R_X(t_1,t_2)}{\partial t_2} = \alpha \quad \text{if } t_2 < t_1$$

$$= 0 \quad t_1 < t_2$$

which has a shape as a function of t_1



Therefore

$$\frac{\partial R_X(t_1,t_2)}{\partial t_2} = \alpha u(t_1 - t_2)$$

as a function t_1 and t_2 fixed

$$\therefore \frac{\partial R_X(t_1,t_2)}{\partial t_1 \partial t_2} = \alpha \frac{\partial}{\partial t_1} u(t_1 - t_2)$$

$$= \alpha \delta(t_1 - t_2).$$

The generalized m-s derivative of Wiener process is called white noise and from an earlier theorem

$$R_{X'}(t_1,t_2) = \frac{\partial R_X(t_1,t_2)}{\partial t_1 \partial t_2} = \alpha \delta(t_1 - t_2)$$

WSS processes and psd; Wiener-Khinchin Relation

Suppose $X(t)$ is a W.S.S. process with

$$R_{XX}(\tau) \triangleq R_{XX}(\tau, 0) = R_{XX}(t+\tau, t) \quad \forall t$$

WIENER-KHINCHIN Relationship.

Then define

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

i.e. $S_{XX}(\omega) = FT(R_{XX})$

and

$$R_{XX}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega t} d\omega$$

i.e. $R_{XX} = IFT(S_{XX})$

Now suppose $Y(t) = L[X(t)]$ defined by

$$Y(t, \xi) = \int_{-\infty}^{\infty} h(\tau) X(t-\tau, \xi) d\tau$$

Power Spectral density

$$R_{YX}(\tau) = E[\,Y(t+\tau)\,X^*(t)\,]$$

$$= E\left[\int_{-\infty}^{\infty} h(t+\tau-\alpha)\,X(\alpha)\,d\alpha\,X^*(t)\right]$$

$$= \int_{-\infty}^{\infty} h(t+\tau-\alpha)\,E(X(\alpha)X^*(t))\,d\alpha$$

$$= \int_{-\infty}^{\infty} h(t+\tau-\alpha)\,R_{XX}(t-\alpha)\,d\alpha$$

$$= \int_{-\infty}^{\infty} h(\tau-\alpha')\,R_{XX}(\alpha')\,d\alpha'$$

$$= h(\tau) * R_{XX}(\tau)$$

Power spectral density

Then

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

Now,

$$R_{YY}(\tau) = E\left[Y(t+\tau)\, Y^*(t)\right]$$

$$= E\left[Y(t+\tau) \int_{-\infty}^{\infty} X^*(t-\alpha)\, h^*(\alpha)\, d\alpha\right]$$

$$= \int_{-\infty}^{\infty} E\left[Y(t+\tau)\, X^*(t-\alpha)\right] h^*(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} R_{YX}(\tau+\alpha)\, h^*(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} R_{YX}(\tau-\beta)\, h^*(-\beta)\, d\beta$$

$$= R_{YX}(\tau) * h^*(-\tau)$$

We have shown that

$$R_{YX}(\tau) = h(\tau) * R_{XX}(\tau)$$

$$\therefore\ R_{YY}(\tau) = h(\tau) * R_{XX}(\tau) * h^*(-\tau)$$

Power Spectral Density

$$\therefore \quad S_{YY}(\omega) = H(\omega)\, S_{XX}(\omega)\, H^*(\omega)$$

$$= |H(\omega)|^2\, S_{XX}(\omega)$$

Thus

$$S_{XX}(\omega) \longrightarrow \boxed{|H(j\omega)|^2} \longrightarrow S_{YY}(\omega).$$

if

$$X \longrightarrow \boxed{h} \longrightarrow Y$$

$$Y = h * X.$$

White noise input

Evidently if the input is white noise

$$R_X(\tau) = \delta(\tau)$$

$$\Rightarrow \quad S_{XX}(\omega) = 1$$

and

$$S_{YY}(\omega) = |H(\omega)|^2$$

—— x ——.

Why is it power Spectral density

Theorem: Let $X(t)$ be a stationary random process with finite variance, autocorrelation function $R(\tau)$ and power Spectral density $S_X(\omega)$. Then $S_X(\omega) \geq 0$ and for $\omega_2 \geq \omega_1$

$$\frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_X(\omega)\, d\omega$$ is the average power of $X(t)$ in the frequency band (ω_1, ω_2)

Proof: Define a filter H such that

$$H(\omega) = 0 \quad \text{if} \quad \omega \notin [\omega_1, \omega_2]$$
$$= 1 \quad \text{if} \quad \omega \in [\omega_1, \omega_2]$$

Suppose $Y(t)$ is the output of the filter with $X(t)$ the input. Then from the theorem we have proven

Why is it psd

$$S_{YY}(\omega) = S_{XX}(\omega) \quad \text{if} \quad \omega \in [\omega_1, \omega_2]$$
$$= 0 \quad \text{otherwise.}$$

and the autocorrelation function

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, e^{j\omega\tau}\, d\omega$$

and in particular

$$0 \leq E[Y(t)Y(t)] = R_Y(0)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega$$
$$= \frac{1}{2\pi}\int_{+\omega_1}^{\omega_2} S_{XX}(\omega)\, d\omega$$

$\therefore$ The output of a filter that retains only the frequencies between (ω_1, ω_2)

has average power
$$\frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_{XX}(\omega)\, d\omega$$

In particular the average power in the

why call it psd

the frequency range $(\omega, \omega+\Delta\omega)$ is

$$\frac{1}{2\pi}\int_{\omega}^{\omega+\Delta\omega} S_{xx}(\omega')\,d\omega' \geq 0$$

$$\simeq \frac{1}{2\pi} S_{xx}(\omega)\,\Delta\omega \geq 0$$

$\therefore \quad S_{xx}(\omega) \geq 0$

and the power density of $x(t)$

at the frequency ω is

$S_{xx}(\omega)$.

Δ

Remark:

One can start with the definition of $S_{xx}(\omega)$ as the average power of x in the frequency range $(\omega, \omega+\Delta\omega)$ and prove that

$S_{xx}(\omega)$ is the Fourier transform of the autocorrelation function. This what is done in physics books and is called the Wiener-Khinchin theorem

Examples

Consider the stochastic process generated by

$$Y(t+dt) - Y(t) = -\alpha Y(t)\,dt + \underbrace{\sqrt{dt}\;N(0,dt)}_{dW_t}$$

wiener process

$\Rightarrow$ Symbolically this can be rewritten as

$$\frac{Y(t+dt) - Y(t)}{dt} = -\alpha Y(t) + \frac{dW_t}{dt}$$

$$\Rightarrow \quad \frac{dY(t)}{dt} = -\alpha Y(t) + n(t)$$

where $n(t)$ is white noise (the derivative of wiener process)

$$\Rightarrow \quad \frac{dY(t)}{dt} + \alpha Y(t) = n(t);$$

Assume $\alpha > 0$.

Then $n(t)$ is white noise.

Note that we have shown earlier that the wiener process

Examples

has mean $\langle W(t)\rangle = 0$

and variance $\langle W^2(t)\rangle = t$

and $R_{WW}(t_1,t_2) = \min(t_1,t_2)$.

$\therefore$ as n is the derivative of the Wiener process we have

$$\langle n(t)\rangle = \frac{d}{dt}\langle W(t)\rangle$$
$$= 0$$

and $R_{nn}(t_1,t_2) = \delta(t_1-t_2)$

↑ established earlier.

Thus, $n(t)$ is wide sense stationary with mean 0 and autocorrelation

$$R_n(\tau) = \delta(\tau).$$

Coming back to the stochastic differential equation

$$\frac{dY(t)}{dt} + \alpha Y(t) = n(t)$$

Example

The filter corresponding to the above equation is

$$n(t) \longrightarrow \underbrace{\boxed{\frac{1}{s+\alpha}}}_{H} \longrightarrow y(t)$$

and in the fourier domain

$$H(j\omega) = \frac{1}{\alpha + j\omega}$$

$$\Rightarrow H(j\omega) = \frac{\alpha - j\omega}{\alpha^2 + \omega^2}$$

$$= \frac{\alpha}{\alpha^2 + \omega^2} - j\frac{\omega}{\alpha^2 + \omega^2}$$

$$\Rightarrow |H(j\omega)|^2 = \frac{\alpha^2}{(\alpha^2 + \omega^2)^2} + \frac{\omega^2}{(\alpha^2 + \omega^2)^2}$$

$$= \frac{1}{(\alpha^2 + \omega^2)}$$

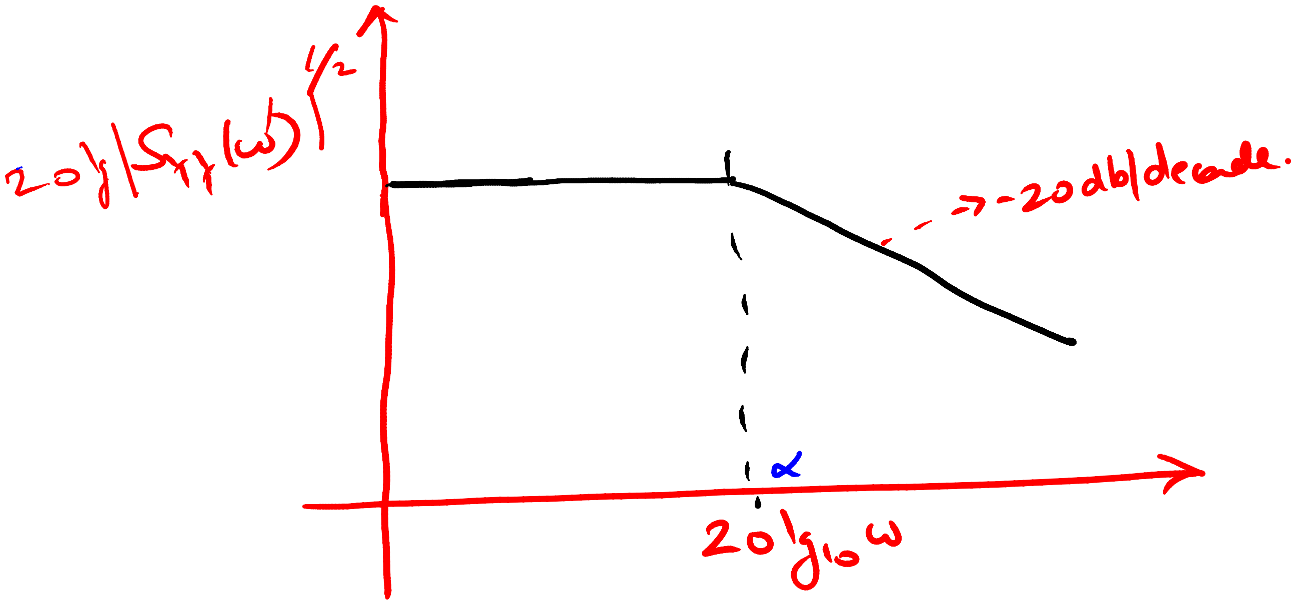
Now, $S_{nn}(\omega) = 1$

$$\therefore S_{yy}(\omega) = |H(j\omega)|^2 S_{nn}(\omega) = \frac{1}{\alpha^2 + \omega^2}$$

Example

Thus, Y has a spectral density

$$S_{YY}(\omega) = \frac{1}{\alpha^2+\omega^2} = \frac{1}{\alpha^2}\frac{1}{(\frac{\omega}{\alpha})^2+1}$$



$$20\log_{10}|S_{YY}(\omega)|^{1/2} = +20\lg_{10}\sqrt{\frac{1}{\alpha^2((\omega/\alpha)^2+1)}}$$

$$= 20\lg_{10}\frac{1}{\alpha} - 20\lg\sqrt{(\omega/\alpha)^2+1}$$

$\therefore$ for $\omega \ll \alpha$

$$20\lg_{10}|S_{YY}(\omega)|^{1/2} = -20\lg\alpha$$

$\omega > \alpha$ $\quad 20\lg_{10}|S_{YY}(\omega)|^{1/2} = -20\lg\alpha - \underbrace{20}_{\text{slope}}\lg\omega$

α can be easily determined from the corner frequency.