EE 5505 Prof. Jindal
Wireless Communication February 13, 2010

Homework 3

Due: Friday, February 19, 4:00 PM

Please turn in your MATLAB scripts in addition to your solutions and plots.

1. Consider the general tap-delay model

$$h(\tau; t) = \sum_{i=1}^{L} g_i(t)\delta(\tau - \tau_i), \tag{1}$$

where the coefficients $g_1(t), \ldots, g_L(t)$ are independent and each is complex Gaussian with zero mean and $E[|g_i(t)|^2] = \alpha_i$ for $i = 1, \ldots, L$.

The frequency response with respect to τ is given by (we derived this in lecture):

$$C(f; t) = \int_{-\infty}^{\infty} c(\tau, t)e^{-j2\pi f\tau}d\tau \tag{2}$$

$$= \sum_{i=1}^{L} g_i(t)e^{-j2\pi f\tau_i} \tag{3}$$

(a) Show that $C(f; t)$ is complex Gaussian.

(b) Compute the mean and variance of $C(f; t)$

Hint: Multiplying a complex Gaussian random variable (with iid real and imaginary parts, as we consider in this model) by a phase term (e.g., $e^{j\theta}$) does not change the distribution of the complex Gaussian.

2. In this problem we will simulate a wideband channel according to a 3 tap model:

$$c(\tau, t) = \sum_{i=1}^{3} g_i(t)\delta(\tau - \tau_i). \tag{4}$$

Let us assume the typical model, as discussed in class (Rayleigh tap coefficients, with temporal correlation given by the Jakes model):

- Delays: $\tau_1 = 0$, $\tau_1 = 1$ μsec, $\tau_2 = 3$ μsec
- Stationary distribution: Each of the tap coefficients $g_i(t)$ are complex Gaussian (i.e., Rayleigh fading) and are independent, with powers given by: $\alpha_1 = E[|g_1(t)|^2] = 1$, $\alpha_2 = E[|g_2(t)|^2] = 0.5$, and $\alpha_3 = E[|g_3(t)|^2] = 0.1$.
- Temporal correlation: The tap coefficients $g_i(t)$ are each independent Gaussian processes, with temporal correlation as described by the Jakes model, with Doppler frequency of 100 Hz.

You can simulate a channel of the form of (4) by simulating each of the tap coefficients $g_i(t)$ in time (t). To assist you with this, a Matlab function that simulates Jakes (by performing filtering in the frequency domain) is provided to you.

(a) Compute the coherence time and the coherence bandwidth.

(b) Plot the power (dB units) of the tap coefficients $g_1(t), g_2(t)$ and $g_3(t)$ versus time t for a period of 0.1 seconds. All three should be on a single plot.

(c) Perform a Fourier transform of $c(\tau, t)$ (with respect to τ) for $t = 0$, $t = 0.001$, and $t = 0.1$ seconds, and plot the three frequency responses on the same figure (these correspond to $C(f;t)$). Explain why you would expect the frequency response at $t = 0$ and $t = 0.001$ to be very similar, while the frequency response at $t = 0.1$ is quite different.

3. In this problem we will derive the probability of bit error for QPSK in Rayleigh fading. Recall from class that the probability of bit error of QPSK in AWGN is $Q(\sqrt{\text{SNR}})$, while in Rayleigh fading it is:

$$P_{\text{bit error}} = \mathbb{E}_{|h|^2}\left[Q\left(\sqrt{|h|^2\text{SNR}}\right)\right] \tag{5}$$

$$= \int_0^\infty Q\left(\sqrt{x \cdot \text{SNR}}\right) f_{|h|^2}(x)dx \tag{6}$$

where $f_{|h|^2}(x) = e^{-x}$ is the PDF of the fading power $|h|^2$.

(a) Show

$$P_{\text{bit error}} = \frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{2 + \text{SNR}}}\right)$$

by writing (6) as a double integral, changing the order of integration, and then evaluating the integral.

(b) The bit error expression can alternatively be expressed in terms of the CDF of a random variable that follows the F-distribution. First show that the bit error expression can alternatively be written as:

$$P_{\text{bit error}} = \frac{1}{2}\mathbb{P}\left[\frac{|h|^2}{w^2} \leq \frac{1}{\text{SNR}}\right]$$

where w is real and is $N(0,1)$. Then verify that random variable $\frac{|h|^2}{w^2}$ follows the F-distribution with parameters $(2, 1)$ (you should be able to easily find information about the F-distribution and the chi-square distribution on the web or in probability textbooks), and finally that the CDF of the F-distribution gives the correct formula.

(c) Perform a first-order Taylor expansion about the point $x = 0$ to show

$$\sqrt{\frac{1}{1+x}} \approx 1 - \frac{x}{2}$$

for $x \approx 0$. Then use this approximation to argue

$$P_{\text{bit error}} \approx \frac{1}{2\text{SNR}}$$

for large values of SNR.

4. In this problem we study the *block* error rate when using uncoded QPSK, and compare our results to capacity. We transmit a block of n QPSK symbols, and say a block error occurs if one or more of the n symbols are detected incorrectly.

   (a) Show that the probability of a block error is:

   $$P_{\text{block}} = 1 - (1 - Q(\sqrt{\text{SNR}}))^{2n}.$$

   (b) Plot block error probability versus SNR for SNR between 0 and 15 dB for $n = 10$, $n = 100$, $n = 500$, and $n = 1000$. All 4 curves should be on the same plot. SNR should be in dB and the y-axis (prob. error) should be in a logarithmic scale (similar to the figures in Ch. 6).

   (c) According to capacity, what SNR is required to achieve reliable (i.e., block error rate going to zero) communication at a rate of 2 bits/symbol over an AWGN channel?

   (d) A reasonable block error probability is 10^{-2}. From your graph from part (a), write down the SNR required for a block error probability of 10^{-2} for the different block sizes in your plot. How do these required SNR's compare to your answer to (c)?

5. In the block fading model (Rayleigh) with L iid blocks, the outage probability is given by:

   $$P_{out}(R) \quad = \quad \mathbb{P}\left[\frac{1}{L}\sum_{i=1}^{L}\log_2(1 + |h_i|^2\text{SNR}) < R\right]$$

   where $|h_1|^2, \ldots, |h_L|^2$ are iid unit-mean exponentials.

   (a) Show that the following is an upper bound to the outage probability:

   $$P_{out}(R) \leq \left(\mathbb{P}\left[\log_2(1 + |h_1|^2\text{SNR}) < RL\right]\right)^L$$

   (b) Show that the following is a lower bound to the outage probability:

   $$P_{out}(R) \geq \left(\mathbb{P}\left[\log_2(1 + |h_1|^2\text{SNR}) < R\right]\right)^L$$

   (c) Using the approximation $1 - e^{-x} \approx x$ and the bounds, argue that

   $$P_{out}(R) \approx \text{SNR}^{-L}$$

   for large values of SNR.