EE 5505 Prof. N. Jindal
Wireless Communication April 1, 2010

Homework 7
Due: Thursday, April 8, 11:15 AM

1. In this problem we will compare the performance of using a bank of matched filter (MF, or MRC), zero forcing (ZF), and MMSE filters (without successive interference cancellation) for a $n_T = n_R = N$ channel.

   (a) The general formula for the SINR after correlating with linear filter $\mathbf{v}_i$ is:

   $$\frac{(\mathrm{SNR}/N)|\mathbf{v}_i^H \mathbf{h}_i|^2}{(\mathrm{SNR}/N)\sum_{j\neq i}|\mathbf{v}_i^H \mathbf{h}_j|^2 + \|\mathbf{v}_i\|^2}$$

   If we define $\mathbf{K}_i$ as the covariance matrix of the interference (from the perspective of the i-th stream):

   $$\mathbf{K}_i \triangleq \mathbf{I} + \frac{\mathrm{SNR}}{N}\sum_{j\neq i}\mathbf{h}_j\mathbf{h}_j^H$$

   (in this expression we have divided out a factor of N_0 for convenience), show that the SINR can alternatively be written as:

   $$\frac{\mathrm{SNR}}{N}\frac{\mathbf{v}_i^H\left(\mathbf{h}_i\mathbf{h}_i^H\right)\mathbf{v}_i}{\mathbf{v}_i^H\mathbf{K}_i\mathbf{v}_i}.$$

   (b) The matched filter is in the direction of the channel, i.e., $\mathbf{v}_i = \mathbf{h}_i/\|\mathbf{h}_i\|$. Explain why the distribution of the SINR for this filter is the same as:

   $$\frac{(\mathrm{SNR}/N)\chi^2_{2N}}{(\mathrm{SNR}/N)\chi^2_{2(N-1)}+1}.$$

   Based on this equation, argue that the SINR converges to $N/(N-1)$ as $\mathrm{SNR}\to\infty$.

   (c) The MMSE filter (unnormalized) is given by:

   $$\left(\mathbf{I} + \frac{\mathrm{SNR}}{N}\sum_{j\neq i}\mathbf{h}_j\mathbf{h}_j^H\right)^{-1}\mathbf{h}_i.$$

   Based on this expression, argue that the MMSE filter converges to the matched filter as $\mathrm{SNR}\to 0$ and that the MMSE filter converges to the zero-forcing filter (i.e., the filter chosen orthogonal to the channels corresponding to the other data streams) as $\mathrm{SNR}\to\infty$.

(d) Show that the SINR for the MMSE filter is:

$$\frac{\text{SNR}}{N}\mathbf{h}_i^H\left(\mathbf{I}+\frac{\text{SNR}}{N}\sum_{j\neq i}\mathbf{h}_j\mathbf{h}_j^H\right)^{-1}\mathbf{h}_i.$$

Hint: you can use the fact that $\mathbf{K}^{-1}$ is Hermitian, i.e., $(\mathbf{K}^{-1})^H = \mathbf{K}^{-1}$.

(e) Write a Matlab program that uses Monte Carlo simulation to compute the expected rate, i.e., $N\mathbb{E}[\log_2(1+\text{SINR})]$, for MF, ZF, and MMSE. For $N = 4$ plot the expected rate versus SNR (in dB) for SNR going from -10 to 30 dB. Comment on the behavior of MMSE at low and high SNR, based upon (c).

2. In this problem we will quantify the benefit of SIC (successive interference cancellation) at high SNR, again for a $n_T = n_R = N$ channel. Although we are really interested in comparing MMSE and MMSE-SIC, as noted in the previous problem, the MMSE filter converges to the ZF filter at high SNR. Thus, the performance of ZF and MMSE are equivalent at high SNR, as are ZF-SIC and MMSE-SIC. The ZF filter is easier to analyzing, and thus we will compare ZF to ZF-SIC.

If we use a bank of ZF filters without SIC, recall that the effective signal-to-noise ratio for each stream is $(\text{SNR}/N)\chi_2^2$. if we use ZF with SIC the effective signal-to-noise ratio for the j-th stream is $(\text{SNR}/N)\chi_{2j}^2$; SIC reduces the interference to subsequent streams, and thus streams that are decoded later have a larger SNR. We are interested in comparing the average rate achieved by each strategy, where the corresponding rates are:

$$\begin{aligned} R_{\text{ZF}}(\text{SNR}, N) &= N\mathbb{E}\left[\log_2\left(1+\frac{\text{SNR}}{N}\chi_2^2\right)\right] \\ R_{\text{ZF-SIC}}(\text{SNR}, N) &= \sum_{j=1}^{N}\mathbb{E}\left[\log_2\left(1+\frac{\text{SNR}}{N}\chi_{2j}^2\right)\right] \end{aligned}$$

(a) The high-SNR offset is defined as the difference between the two rates in the limit $\text{SNR}\to\infty$:

$$\Delta_N \triangleq \lim_{\text{SNR}\to\infty}\left[R_{\text{ZF-SIC}}(\text{SNR}, N) - R_{\text{ZF}}(\text{SNR}, N)\right]$$

Using the fact that one plus term in the $\log(\cdot)$ expression can be neglected at high SNR, and the property

$$\mathbb{E}\left[\log_2\left(\chi_{2j}^2\right)\right] = \left(\psi(0)+\sum_{l=1}^{j-1}\frac{1}{l}\right)\log_2 e$$

where $\psi(0)$ is Euler's constant, find an expression for Δ_N.

(b) The expression for Δ_N in the previous part was expressed in spectral efficiency units (i.e., bps/Hz), but we are often interested in converting these quantities to power units (dB). For single antenna systems we had the rule of thumb that 1 bps/Hz is equal to 3 dB, whereas for MIMO systems with N transmit and receive antennas, each 3 dB of power provides an additional N bps/Hz. Convert Δ_N to power units, and comment on whether the power penalty (of ZF relative to ZF-SIC) is increasing or decreasing with N. Explain.

(c) Use Matlab (and Monte Carlo simulation) produce a plot of $R_{\text{ZF}}(\text{SNR}, N)$ and $R_{\text{ZF-SIC}}(\text{SNR}, N)$ versus SNR for $N = 2$ and $N = 8$ (on same plot). At what SNR does the power offset computed in part (b) begin to accurately represent the difference between ZF and ZF-SIC.

3. There are a number of analytical results in the *large-system limit*, which refers to keeping SNR fixed while increasing the number of antennas N (with $N = n_T = n_R$). As we discussed in class, for SNR fixed and $N \to \infty$ we have:

$$\begin{aligned} R_{\text{ZF}}(\text{SNR}, N) &\to \text{SNR} \log_2(e) \\ R_{\text{ZF-SIC}}(\text{SNR}, N) &\to N \left[\left(1 + \frac{1}{\text{SNR}}\right) \log_2(1 + \text{SNR}) - \log_2(e) \right] \end{aligned}$$

(the first result is exact, whereas the second was derived based on an upper bound to $R_{\text{ZF-SIC}}$). Researchers have also shown:

$$R_{\text{MMSE}}(\text{SNR}, N) \to N \log_2 \left(1 + \sqrt{\text{SNR} + \frac{1}{4}} - \frac{1}{2} \right)$$

From problems 1 and 2 we know that, for fixed N, there is only a constant difference between R_{ZF}, $R_{\text{ZF-SIC}}$, and R_{MMSE} as $\text{SNR} \to \infty$. This would seem to indicate that ZF, ZF-SIC, and MMSE have somewhat similar behavior. However, the large-system results show that ZF is very inferior compared to ZF-SIC and MMSE. Explain why the large system results are not inconsistent with the high SNR results from the earlier problems.