EE 5581 — Prof. Jindal
Information Theory & Coding — Oct. 31, 2005

Homework 7

Due: Tuesdsay, November 8

In the following problems we will show that capacity is the fundamental limit on communication even if the performance criterion is bit error rate instead of block error rate. In lecture we were interested in systems that had block error rate P_B converging to zero as block length (n) went to infinity. However, in many practical systems we are interested in the bit error rate P_b.

Consider a system where the message W is an equiprobable k-bit sequence, denoted $U_1, \ldots, U_k$, and the output of the encoder is an n-bit sequence denoted $X_1, \ldots, X_n$. Clearly the rate of our code is $R = k/n$. The output of the channel is $Y_1, \ldots, Y_n$. The decoder outputs an estimate of the message bits denoted $\hat{U}_1, \ldots, \hat{U}_k$.

The expressions for block (P_B) and bit (P_b) error rate are:

$$P_B = \Pr\{(\hat{U}_1, \ldots, \hat{U}_k) \neq (U_1, \ldots, U_k)\} \qquad P_b = \frac{1}{k}\sum_{i=1}^{k} \Pr\{\hat{U}_i \neq U_i\} = \frac{1}{k}\sum_{i=1}^{k} P_{e,i}$$

where $P_{e,i} = \Pr\{\hat{U}_i \neq U_i\}$.

1. Prove $H(p)$ is a concave function of p (without using the convexity of the K-L distance). Note that p here represents a vector, i.e., a probability distribution.

2. (a) Show that P_b and P_B satisfy:

$$P_b \leq P_B \leq kP_b.$$

   (b) Use Fano's inequality to prove

$$H(U_i|Y^n) \leq h(P_{e,i})$$

   where $h(\cdot)$ is the binary entropy function.

   (c) Prove the following:

$$\frac{1}{k}\sum_{i=1}^{k} h(P_{e,i}) \leq h(P_b)$$

3. (a) Justify steps (a)-(g) in the following proof:

$$
\begin{aligned}
k = H(U^k) &\overset{(a)}{=} I(U^k;Y^n) + H(U^k|Y^n) \\
&\overset{(b)}{\leq} I(X^n;Y^n) + H(U^k|Y^n) \\
&\overset{(c)}{\leq} \sum_{i=1}^{n} I(X_i;Y_i) + H(U^k|Y^n) \\
&\overset{(d)}{\leq} nC + H(U^k|Y^n) \\
&\overset{(e)}{\leq} nC + \sum_{i=1}^{k} H(U_i|Y^n) \\
&\overset{(f)}{\leq} nC + \sum_{i=1}^{k} h(P_{e,i}) \\
&\overset{(g)}{\leq} nC + kh(P_b)
\end{aligned}
$$

(b) How can we rearrange the final inequality in the previous part to get:

$$h(P_b) \geq 1 - \frac{C}{R}$$

By taking the inverse of both sides we get:

$$P_b \geq h^{-1}\left(1 - \frac{C}{R}\right).$$

Note that the inverse of the binary entropy function actually has two values, but only the smaller of the two is relevant for the lower bound on P_b.

(c) Why does the previous part imply that $P_b > 0$ if $R > C$? Is this true for all blocklengths?

4. The above problem indicates that we can actually transmit at rates strictly above capacity if we are willing to allow a non-zero bit error rate.

   (a) If $R = 1.1C$, what is the minimum possible bit error rate? What about at $R = 1.01C$?

   (b) At what rate can we transmit if allow a bit error rate of 10^{-3}? How about 10^{-4}?

5. How can the converse be modified to work if feedback is allowed, i.e. the i-th channel input is allowed to be a function of $U_1, \ldots, U_k$ and the previous channel outputs $y_1, \ldots, y_{i-1}$? Write down the full converse, referring back to previous problems where appropriate.