Feedback capacity of multiple access & broadcast channels

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Outline

- Single-user capacity with feedback
- Multiple-access channel (MAC)
  - Achievable region of Cover & Leung
    - Pf: Block Markov Coding (Random Coding)
  - Capacity not known – an outer bound
  - White Gaussian MAC
    - For 2 Tx capacity is found by Ozarow
      - Pf: Deterministic coding of Schalkwijk & Kailath

Outline 2

- Broadcast Channel (BC)
  - Physically degraded BC
    - No capacity increase with feedback
  - Stochastically degraded BC
    - White Gaussian BC
      - Achievable region of Ozarow & Leung
      - Capacity not known – an outer bound

Note ...

- Feedback is instantaneous & noiseless
  - May not be practical
  - Noiseless feedback: Satellite links
  - Upper bound on practical feedback

Single user results

- Discrete memoryless channel
  - Capacity not changed by feedback: Shannon
  - Same result for white Gaussian channel
- Gaussian channel with correlated noise
  - Capacity is at most doubled by feedback
    - Proof by Pinsker and Ebert
  - Capacity is at most increased by half a bit
    - Proof by Cover and Pombra

MAC without feedback

- Capacity region:

$$R_1 \leq I(X_1; Y | X_2)$$

$$R_2 \leq I(X_2; Y | X_1)$$

$$R_1 + R_2 \leq I(X_1; X_2; Y)$$

Where $p(x_1; x_2; y) = p(y|x_1; x_2)p(x_1)p(x_2)$

Any point in the convex hull can be achieved by time sharing

Achievable region of Cover & Leung

$$
\begin{aligned}
R_1 &< I(X_1; Y | X_2; U) \\
R_2 &< I(X_2; Y | X_1; U) \\
R_1 + R_2 &< I(X_1; X_2; Y)
\end{aligned}
$$

Where the cardinality of U is bounded as:

$$||U|| = \min\{||X_1|| \times ||X_2||; ||Y||\}$$

And: $p(u; x_1; x_2; y) = p(u)p(x_1|u)p(x_2|u)p(y|x_1; x_2)$

Proof: Block Markov Coding

Block Markov Coding

- Tx B blocks of length n each
- Tx:
  - At block b superimpose & send
    - New data
    - An index to help Rx decode block b-1
  - Decode other Tx message by feedback
- Rx:
  - Decodes block b-1 using index of block b
  - Limits the possible estimates for block b

Block Markov Coding 2

$w_1 \in \{1, \cdots, 2^{nR_1}\}$ $w_2 \in \{1, \cdots, 2^{nR_2}\}$ $j \in \{1, \cdots, J\}$

Choose $u^n(j)$: $\prod_{i=1}^{n} p(u_i)$. This codebook is the same for both Tx.

Tx1: Choose $x_1^n(w_1, j)$: $\prod_{i=1}^{n} p(x_{1i} | u_i(j))$. Send $x_1^n(w_1, j)$ at block b.

Tx2: Choose $x_2^n(w_2, j)$: $\prod_{i=1}^{n} p(x_{2i} | u_i(j))$. Send $x_2^n(w_2, j)$ at block b.

Rx: Find unique j s.t. $(u^n(j), y^n) \in A_\epsilon^n$

Find all pairs (w_1, w_2) s.t. $(x_1^n(w_1, j), x_2^n(w_2, j), y^n) \in A_\epsilon^n$

Tx1: Find unique w_2 s. t. $(x_1^n(w_1, j), x_2^n(w_2, j), y^n) \in A_\epsilon^n$

Tx2: Find unique w_1 s. t. $(x_1^n(w_1, j), x_2^n(w_2, j), y^n) \in A_\epsilon^n$

An Outer Bound ...

$$R_1 \leq I(X_1;Y|X_2)$$

$$R_2 \leq I(X_2;Y|X_1)$$

$$R_1 + R_2 \leq I(X_1,X_2;Y)$$

Let C_0 denote the convex hull over all rate pairs (R_1,R_2) satisfying the above equations for a given $p(x_1,x_2)$.

$$C_{fb} \subseteq C_0$$

White Gaussian MAC: no feedback

For two Tx with average power constraints:

$$E[X_1^2] \cdot P_1; \qquad E[X_2^2] \cdot P_2$$

$$R_1 \cdot \frac{1}{2}\log(1+\frac{P_1}{N})$$

$$R_2 \cdot \frac{1}{2}\log(1+\frac{P_2}{N})$$

$$R_1 + R_2 \cdot \frac{1}{2}\log(1+\frac{P_1+P_2}{N})$$

White Gaussian MAC with feedback

- For 2 Tx capacity found by Ozarow:

$$R_1 \le \frac{1}{2}\log\left(1 + \frac{P_1}{N}(1-\rho^2)\right)$$

$$R_2 \le \frac{1}{2}\log\left(1 + \frac{P_2}{N}(1-\rho^2)\right)$$

$$R_1 + R_2 \le \frac{1}{2}\log\left(1 + \frac{P_1 + P_2 + 2\rho\sqrt{P_1P_2}}{N}\right)$$

Capacity region is the convex hull of all rate pairs $(R_1; R_2)$
Satisfying above equations for a ρ: $0 \le \rho \le 1$.

Deterministic Coding with feedback

- Proposed by Schalkwijk & Kailath
  - Map messages to real numbers in [-.5,.5]
  - Send that number instead of the message
  - Rx feeds back an estimate of that number
  - After receiving the feedback
    - Tx sends the error in Rx estimate
    - Rx updates its estimate at each iteration
    - Rx feeds back its new estimate
    - Repeat until error variance is small enough
    - Closest value to Rx final estimate is chosen

Deterministic Coding with feedback 2

Tx i has message $m_i \in \{0, \cdots, M_i - 1\}$ to transmit for $i = 1, 2$.

Tx i: $\theta_i = \frac{m_i}{M_i - 1} - \frac{1}{2}$ $\qquad \theta_i \sim U[-0.5, 0.5]$

At time $k = 1$: Tx 1 sends $\sqrt{12P_1}\theta_1$ while Tx 2 is quiet

At time $k = 2$: Tx 2 sends $\sqrt{12P_2}\theta_2$ while Tx 1 is quiet

Rx: At time $k = 1$ receives $Y_1 = \sqrt{12P_1}\theta_1 + Z_1$

And forms the estimate $\hat{\theta}_1^1 = \hat{\theta}_1^2 = \frac{Y_1}{\sqrt{12P_1}}$

At time $k = 2$ receives $Y_2 = \sqrt{12P_2}\theta_2 + Z_2$ and forms $\hat{\theta}_2^2 = \frac{Y_2}{\sqrt{12P_2}}$

Deterministic Coding with feedback 3

Rx feeds back the its estimates to the transmitters

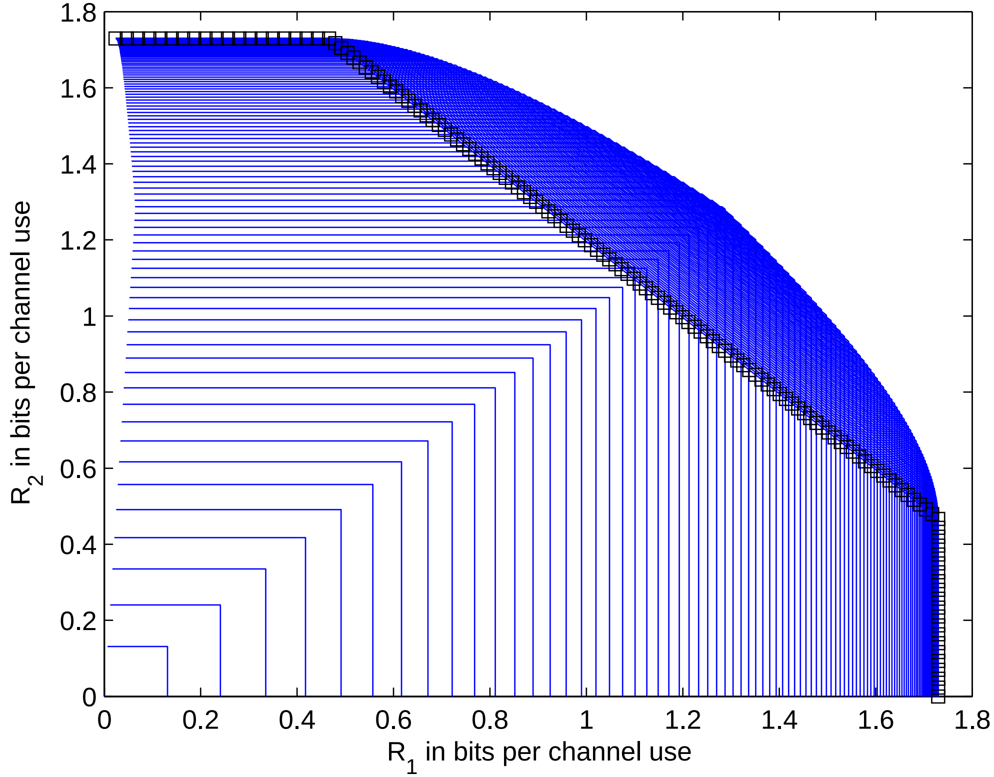
At time $k+1$ where $k \geq 2$:

Tx 1: $X_{1;k+1} = \sqrt{\frac{P_1}{\alpha_{1;k}}}\,\epsilon_{1;k}$ $\qquad \epsilon_{1;k} = \hat{\theta}_1^k - \theta_1$

Tx 2: $X_{2;k+1} = \sqrt{\frac{P_2}{\alpha_{2;k}}}\,\epsilon_{2;k}\,\mathrm{sign}(\rho_k)$ $\qquad \epsilon_{2;k} = \hat{\theta}_2^k - \theta_2$

Rx: Forms the LMMSE $\qquad \hat{\theta}_i^{k+1} = \hat{\theta}_i^k - \frac{\overline{Y_{k+1}\epsilon_{i;k}}}{\overline{Y_{k+1}^2}} Y_{k+1}$

MAC: Feedback Vs. No feedback



2 Tx + 1 Rx

$P_1 = P_2 = 10$

N= 1

Broadcast Channel

- Degraded BC without feedback

$$R_1 \le I(X;Y|U)$$

$$R_2 \le I(U;Z)$$

Where $||U|| \le \min\{||X||, ||Y||, ||Z||\}$

Capacity is the convex hull over all achievable rate pairs for a distribution $p(u,x,y,z) = p(u)p(x|u)p(y,z|x)$

Degraded BC with feedback

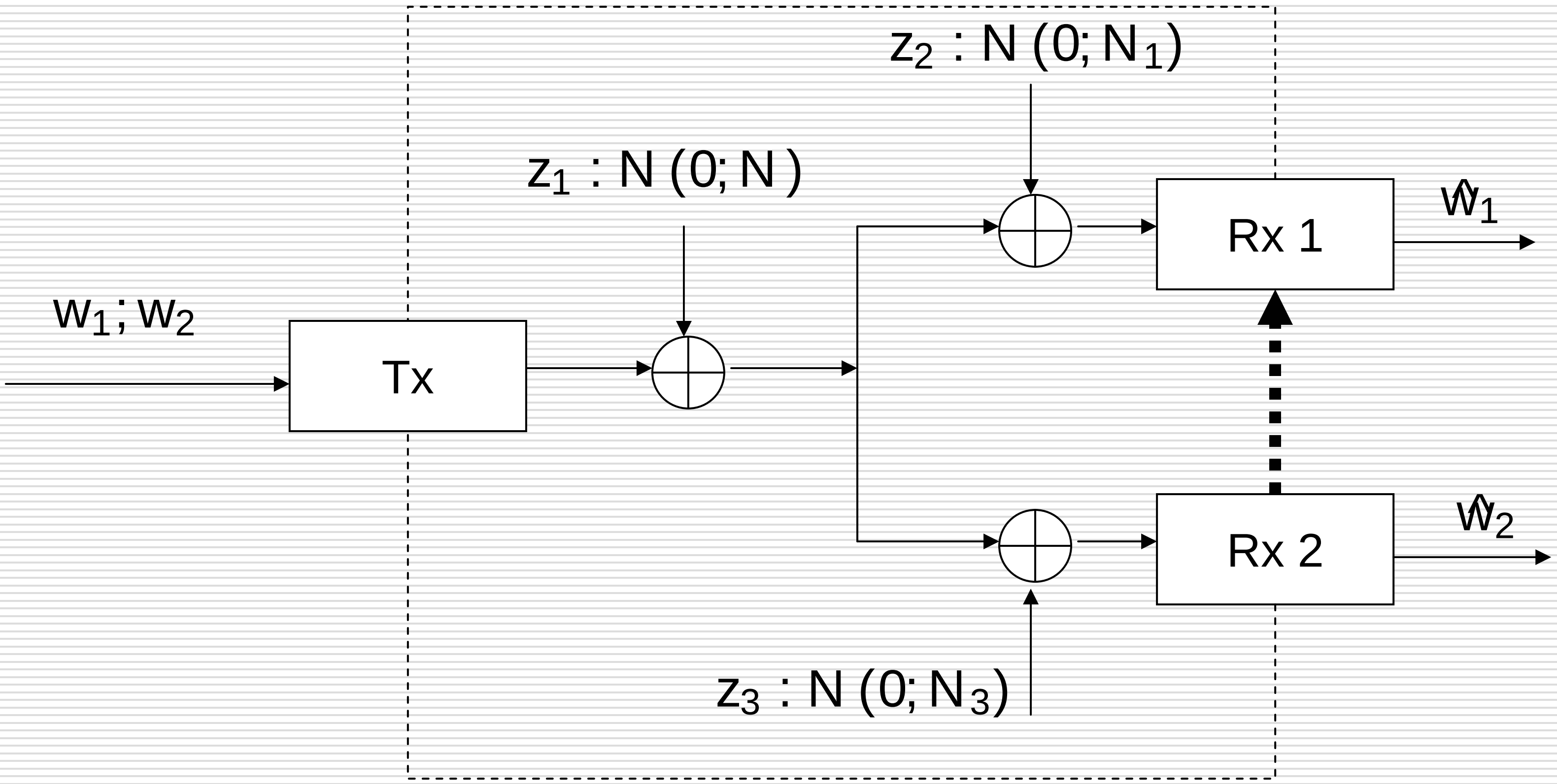
- Physically degraded BC
  - Capacity does not increase with feedback
    - Converse similar to BC without feedback
    - Proved by El Gamal
    - Physically degraded white Gaussian BC
      - Converse proved by El Gamal
      - Uses the modified entropy power inequality

$$e^{\frac{2H(Y)}{n}} \geq e^{\left[\frac{2}{n}\sum_{i=1}^{n} H(X_i \mid Y^{i-1})\right]} + e^{\frac{2H(Z)}{n}}$$

Degraded BC with feedback 2

- Degraded white Gaussian BC
  - Feedback strictly enlarges capacity
  - Achievable region of Ozarow & Leung
    - Proof: Deterministic coding similar to MAC
  - Capacity not known
    - An outer bound by Ozarow & Leung

White Gaussian BC: Model



White Gaussian BC: no feedback

$$R_1 \leq \frac{1}{2}\log\left(1+\frac{\alpha P}{N+N_1}\right)$$

$$R_2 \leq \frac{1}{2}\log\left(1+\frac{(1-\alpha)P}{N+N_2+\alpha P}\right)$$

$\alpha \in [0,1]$ allocates total power between R_1 and R_2

Ozarow & Leung Region

$$R_1 \le \frac{1}{2}\log\left(\frac{N + N_1 + P}{N + N_2 + \frac{P}{D^*}g^2(1-\rho^{*2})}\right)$$

$$R_2 \le \frac{1}{2}\log\left(\frac{N + N_2 + P}{N + N_2 + \frac{P}{D^*}(1-\rho^{*2})}\right)$$

Where $D^* = 1 + g^2 + 2g\rho^*$ and ρ^* is given as a function of $N; N_1; N_2; P; g$

Ex. $P = 10;\ N = 0;\ N_1 = N_2 = 1$

No feedback: $R_1 = R_2 =$ 0.59947 nats

Feedback: $R_1 = R_2 =$ 0.70468 nats

Outer bound: $R_1 = R_2 =$ 0.71956 nats

Outer bound for BC with feedback

- Let Rx1 know the output of Rx2
  - Results in a physically degraded BC
    - Capacity with feedback is known

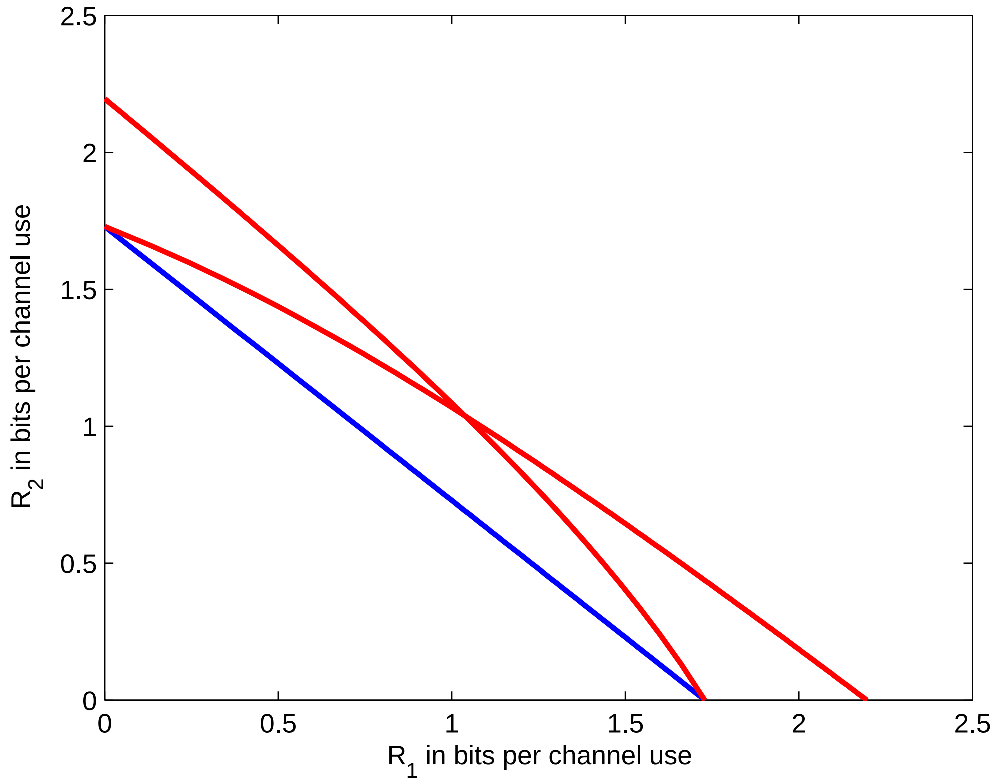
$$R_1 \le \frac{1}{2}\log\left(1+\frac{\alpha P}{N+\frac{N_1 N_2}{N_1+N_2}}\right)$$

$$R_2 \le \frac{1}{2}\log\left(1+\frac{(1-\alpha)P}{N+N_2+\alpha P}\right)$$

Reverse the order of $R_1; R_2$ to get another outer bound

Capacity region is included in the intersection of these two regions

Inner-Outer bounds for BC



P= 10

N= 0

$N_1 = N_2 = 1$

Conclusions

- MAC
  - Feedback strictly enlarges the capacity
  - Only case where capacity is known:
    - White Gaussian MAC with 2 Tx
- BC
  - Physically degraded channels:
    - No increase in capacity
  - White Gaussian BC
    - Feedback strictly enlarges capacity

Thank You!