EE 5581 Prof. N. Jindal
Information Theory & Coding Feb. 1, 2008

Homework 1

Due: Friday, Feb. 8, 5:00 PM

1. (a) Write expressions for the mutual information achieved (per real-dimension) by 2-PAM and 4-PAM in an AWGN channel. Recall that an M-PAM constellation with average energy E has points:

$$\sqrt{\frac{3E}{M^2-1}} \times \{-M+1, -M+3, \ldots, M-1\}.$$

   (b) Using the expressions from part (a), numerically compute the spectral efficiency of BPSK, 4-QAM, and 16-QAM (using MATLAB or some other software package) and create plots of spectral efficiency vs. SNR as well as spectral efficiency vs. $\frac{E_b}{N_0}$. In each of the two plots, also include the capacity curve.

2. If a finite input constellation is used (e.g., 4-QAM) and hard detection is performed on each received symbol, then the channel effectively becomes a discrete-input, discrete-output channel (here we treat the output of the detector as the channel output, i.e., the detector is part of the channel). As we will see in this problem, the achievable mutual information is significantly reduced by the presence of such a detector.

   (a) Instead of peforming hard detection on each complex symbol, one could alternatively perform hard detection *separately* on the real and imaginary components of the received signal. Explain why this is equivalent (in the sense of the achievable mutual information) to performing hard detection on each complex symbol if the input is a standard QAM constellation. Are there complex constellations for which this is not true?

   (b) If 4-QAM is used, show that hard detection converts the channel into a binary symmetric channel with crossover probability $Q\left(\sqrt{SNR}\right)$ (per real dimension), and therefore that the mutual information achieved by 4-QAM (assuming hard detection) is:

$$I(SNR) = 2\left[1 - H\left(Q\left(\sqrt{SNR}\right)\right)\right]$$

   where $H(\cdot)$ is the binary entropy function and $Q(\cdot)$ is the standard Q-function (i.e., $Q(z)$ is the probability a zero-mean, unit variance Gaussian is greater than or equal to z).

   (c) If 4-QAM (i.e., 2-PAM per real dimension) is used, instead of allowing the detector only two output values (per real dim), performance can be improved by allowing it to output one of three possible values (in which case the channel becomes a binary-input, ternary-output channel). By symmetry, the detector should indicate whether the received symbol fell in the interval $(-\infty, -T]$, $(-T, T]$, or (T, ∞) for some appropriately chosen constant $T > 0$. Write an expression (in terms of T and the SNR) for the corresponding mutual information.

(d) Using the expressions derived in the previous parts, numerically compute the spectral efficiency of 4-QAM with standard hard detection and for 4-QAM with three possible detector values (per real dimension). For the ternary-output detector, at each SNR numerically find the value of T that maximizes mutual information (you should be able to do this easily in MATLAB with a simple loop). Generate a spectral efficiency vs. SNR plot and a spectral efficiency vs. $\frac{E_b}{N_0}$ plot (in each plot also include the spectral efficiency of 4-QAM without hard detection as well as the capacity curve).