Fractional Power Control for Decentralized Wireless Networks

Nihar Jindal, Steven Weber, Jeffrey G. Andrews

Abstract

We propose and analyze a new paradigm for power control in decentralized wireless networks, termed fractional power control. Transmission power is chosen as the current channel quality raised to an exponent $-s$, where s is a constant between 0 and 1. Choosing $s = 1$ and $s = 0$ correspond to the familiar cases of channel inversion and constant power transmission, respectively. Choosing $s \in (0, 1)$ allows all intermediate policies between these two extremes to be evaluated, and we see that neither extreme is ideal. We prove that using an exponent of $s^* = \frac{1}{2}$ optimizes the transmission capacity of an ad hoc network, meaning that the inverse square root of the channel strength is the optimal transmit power scaling. Intuitively, this choice achieves the optimal balance between helping disadvantaged users while making sure they do not flood the network with interference.

I. Introduction

Power control is one of the most fundamental adaptive mechanisms available in wireless networks. For a single user fading channel in which the objective is to maximize expected rate, it is optimal to increase transmission power (and rate) as a function of the instantaneous channel quality according to the well-known waterfilling policy. On the other hand, if there is a target rate/SNR, power should be adjusted so that this target level is exactly met; this corresponds to choosing power as a decreasing function of the instantaneous channel quality. In a multi-user network in which users mutually interfere, power control can be used to adjust transmit power levels so that all users simultaneously achieve their target SINR levels. The Foschini-Miljanic algorithm is an iterative, distributed power control method that performs this task assuming that each receiver tracks its instantaneous SINR and feeds back power adjustments to its transmitter [1].

In this paper, we consider *non-iterative* power control algorithms for a multi-user wireless network with mutually interfering users and a common target SINR. Each transmitter knows the channel quality to its intended receiver, but has no knowledge of (potential) interference from other transmitters. On the basis of only this information, each transmitter must determine its power level. While designing such an algorithm a reasonable objective is to maximize the fraction of successful transmissions in the network. One possibility would be to simply have all transmitters use the same power independent of their current channel conditions. In this case the probability that a particular transmitter-receiver pair meets the required SINR threshold depends on the channel between the pair as well as the locations of interfering transmissions, and clearly users with good channel conditions would have a higher probability of success. Another possibility would be to perform channel inversion, where each transmitter chooses its power inversely proportional to its individual channel quality. With this policy transmit-receive pairs with poor channels are no longer disadvantaged, but the increased interference power could potentially knock out many transmissions in the network.

Motivated by the work of [2] in which the concept of *fractional power control* is proposed for cellular networks, we consider fractional power control policies that fall between channel inversion (i.e., full channel compensation) and constant transmission power (no channel compensation). If H is the channel power between the transmitter and receiver, fractional power control refers to using a transmission power of H^{-s} where the exponent s is chosen in $[0, 1]$. Clearly, $s = 0$ implies constant transmit power, whereas $s = 1$ is channel inversion.

We consider a spatially distributed (decentralized) network, representing either a wireless ad hoc network or unlicensed spectrum usage by many nodes (e.g., Wi-Fi or spectrum sharing systems). We consider a network that has the following key characteristics.

The contact author N. Jindal (nihar@umn.edu) is with the Univ. of Minnesota, S. Weber is with Drexel University, J. Andrews is with the University of Texas at Austin. This research was supported by NSF grant no. 0634763 (Jindal), no. 0635003 (Weber), grant no. 0634979 (Andrews), and the DARPA IT-MANET program, grant no. W911NF-07-1-0028 (all authors). Manuscript date: July 2, 2007.

- Each transmitter communicates with a single receiver that is a distance d meters away.
- Channel attenuation is determined by path loss (with exponent α) and a (flat) fading value H.
- All multi-user interference is treated as noise.
- Transmitters do not schedule their transmissions based on their channel conditions or the activities of other nodes.
- Transmitter node locations are modelled by a homogeneous spatial (2-D) Poisson process.

In many settings some of the above assumptions — particularly the fixed transmit distance — are somewhat artificial. They are used to simplify the analysis and discussion in this paper, but each has been shown to preserve the integrity of the conclusions in our prior work, e.g. [3], [4].

Note that in such a decentralized netwok it is quite feasible that signal power between each transmitter-receiver pair can be measured but nothing is known about concurrent transmissions; this may particularly be true in an unscheduled slotted system in which it is not possible to know beforehand which other transmissions will taken place in a particular time slot. In addition, note that it is not always possible (or desirable) to use an iterative power control algorithm such as that of [1] because, for example, a feedback channel with the required latency may not be available or convergence times may be too long relative to a packet duration.

The contributions of the paper are the suggestion of a fractional power control for wireless networks and the derivation of the optimum power control exponent, $s^* = \frac{1}{2}$. Such an approach to power control is shown to greatly increase the transmission capacity of an ad hoc network for small path loss exponents (as $\alpha \to 2$), with more modest gains for higher attenuation channels. The results open a number of possible avenues for future work in the area of power control, and considering the prevalence of power control in practice, carry several design implications.

II. Preliminaries

A. System Model

We consider a set of transmitting nodes at an arbitrary snapshot in time with locations specified by a homogeneous Poisson point process (PPP), $\Pi(\lambda)$, of intensity λ on the infinite two-dimensional plane, $\mathbb{R}^2$. We consider a reference transmitter — receiver pair, where the reference receiver, assigned index 0, is located without loss of generality, at the origin. Let X_i denote the distance of the i-th transmitting node to the reference receiver. Each transmitter has an associated receiver that is assumed to be located a fixed distance d meters away. Let $H_i = H_{i0}$ denote the (random) distance–independent fading coefficient for the channel separating transmitter i and the reference receiver at the origin; let H_{ii} denote the (random) distance–independent fading coefficient for the channel separating transmitter i from its intended receiver. Received power is modeled by the product of the transmission power times the pathloss (with exponent $\alpha > 2$) times the corresponding fading coefficient. Therefore, the (random) SINR at the reference receiver is:

$$\mathrm{SINR}_0 = \frac{P_0 H_0 d^{-\alpha}}{\sum_{i \in \Pi(\lambda)} P_i H_i X_i^{-\alpha} + \eta}, \tag{1}$$

where η is the noise power. Recall our assumption that transmitters may have knowledge of the channel condition, H_{ii}, connecting it with its intended receiver. By exploiting this knowledge, the transmission power, P_i, may depend upon the channel, H_{ii}. This expression can be related to information theory if Gaussian signaling is used, by noting that the mutual information conditioned on the transmitter locations and fading realizations is:

$$I(X_0; Y_0 | \Pi(\lambda), \mathbf{H}, \mathbf{P}) = \log_2(1 + \mathrm{SINR}_0), \tag{2}$$

where $\mathbf{H} = (H_0, H_1, \ldots)$ and $\mathbf{P} = (P_0, P_1, \ldots)$. The Poisson model requires that nodes decide to transmit independently, which corresponds in the above model to slotted ALOHA [5]. A good scheduling algorithm by definition introduces correlation into the set of transmitting nodes, which is therefore not well modelled by a homogeneous PPP. We discuss the implications of scheduling later in the paper.

B. Transmission Capacity

In the outage-based transmission capacity framework, an outage occurs whenever the SINR falls below a prescribed threshold β, or equivalently whenever the instantaneous mutual information falls below $\log_2(1 + \beta)$. Therefore, the system-wide outage probability is

$$q(\lambda) = \mathbb{P}(\mathrm{SINR}_0 < \beta) \tag{3}$$

Because (3) is computed over the distribution of transmitter positions as well as the iid fading coefficients (and consequently transmission powers), it corresponds to fading that occurs on a time-scale that is comparable or slower than the packet duration (if (3) is to correspond roughly to the packet error rate). The outage probability is clearly an increasing function of the intensity λ.

Define $\lambda(\epsilon)$ as the maximum intensity of *attempted* transmissions such that the outage probability is no larger than ϵ, i.e., $\lambda(\epsilon)$ is the unique solution of $q(\lambda) = \epsilon$. The transmission capacity is then defined as $c(\epsilon) = \lambda(\epsilon)(1-\epsilon)b$, which is the maximum density of *successful* transmissions times the spectral efficiency b of each transmission. In other words, transmission capacity is like area spectral efficiency subject to an outage constraint.

Consider a path-loss only environment ($H_i = 1$ for all i) with constant transmission power ($P_i = p$ for all i). Define $\delta = 2/\alpha < 1$. The main result of [3] is given in the following theorem.

Theorem 1 ([3]): **Pure pathloss.** *Consider a network where the SINR at the reference receiver is given by (3) with $H_i = 1$ and $P_i = p$ for all i. Then the following expressions give tight bounds on the outage probability and transmission attempt intensity for λ, ϵ small:*

$$\begin{aligned} q^{\mathrm{pl}}(\lambda) &\geq q_l^{\mathrm{pl}}(\lambda) = 1 - \exp\left\{ -\lambda \pi d^2 \left(\frac{1}{\beta} - \frac{\eta}{p d^{-\alpha}} \right)^{-\delta} \right\}, \\ \lambda^{\mathrm{pl}}(\epsilon) &\leq \lambda_u^{\mathrm{pl}}(\epsilon) = -\log(1-\epsilon) \frac{1}{\pi d^2} \left(\frac{1}{\beta} - \frac{\eta}{p d^{-\alpha}} \right)^{\delta}. \end{aligned} \tag{4}$$

Here *pl* denotes pathloss. The transmission attempt intensity upper bound, $\lambda_u^{\mathrm{pl}}(\epsilon)$, is obtained by solving $q_l^{\mathrm{pl}}(\lambda) = \epsilon$ for λ. These bounds are shown to be tight approximations for small λ, ϵ respectively, which is the usual regime of interest. Note also that $-\log(1-\epsilon) = \epsilon + O(\epsilon^2)$, which implies that transmission density is approximately linear with the desired outage level, ϵ, for small outages. The following corollary illustrates the simplification of the above results when the noise may be ignored.

Corollary 1: When $\eta = 0$ the expressions in Theorem 1 simplify to:

$$\begin{aligned} q^{\mathrm{pl}}(\lambda) &\geq q_l^{\mathrm{pl}}(\lambda) = 1 - \exp\left\{ -\lambda \pi d^2 \beta^{\delta} \right\}, \\ \lambda^{\mathrm{pl}}(\epsilon) &\leq \lambda_u^{\mathrm{pl}}(\epsilon) = -\log(1-\epsilon) \frac{1}{\pi d^2 \beta^{\delta}}. \end{aligned} \tag{5}$$

III. Fractional Power Control

The goal of the paper is to determine the effect that fractional power control has on the transmission capacity upper bound in (4). We first review the key prior result that we will use, then derive the maximum transmission densities λ for different power control policies. We conclude the section by finding the optimal power control exponent s and discussing some implications.

A. Transmission capacity under constant power and channel inversion

In this subsection we restrict our attention to two specific power control strategies: constant power (or no power control) and channel inversion. Under constant power we set $P_i = p$ for all i for some common power level p. Under channel inversion we set $P_i = \frac{p}{\mathbb{E}[H^{-1}]} H_{ii}^{-1}$ for all i. This means that the received signal power is $P_i H_{ii} d^{-\alpha} = \frac{p}{\mathbb{E}[H^{-1}]} d^{-\alpha}$, which is constant for all i. That is, channel inversion compensates for the random channel fluctuations between each transmitter and its intended receiver. Moreover, the expected transmission power is $\mathbb{E}[P_i] = p$, so that the constant power and channel inversion schemes use the same expected power. A main result of [4] is given in the following theorem.

Theorem 2 ([4]): **Constant power.** *Consider a network where the SINR at the reference receiver is given by (3) with $P_i = p$ for all i. Then the following expressions give good approximations of the outage probability and* *transmission attempt intensity for λ, ϵ small:*

$$\begin{aligned}
q^{\mathrm{cp}}(\lambda) &\geq q_l^{\mathrm{cp}}(\lambda) = 1 - \mathbb{E}\left[\exp\left\{-\lambda\pi d^2 \mathbb{E}[H^\delta]\left(\frac{H}{\beta} - \frac{\eta}{p d^{-\alpha}}\right)^{-\delta}\right\}\right] \\
&\approx \tilde{q}_l^{\mathrm{cp}}(\lambda) = 1 - \exp\left\{-\lambda\pi d^2 \mathbb{E}[H^\delta]\mathbb{E}\left[\left(\frac{H}{\beta} - \frac{\eta}{p d^{-\alpha}}\right)^{-\delta}\right]\right\}, \\
\lambda^{\mathrm{cp}}(\epsilon) &\approx \tilde{\lambda}^{\mathrm{cp}}(\epsilon) = -\log(1-\epsilon)\frac{1}{\pi d^2}\frac{1}{\mathbb{E}[H^\delta]}\mathbb{E}\left[\left(\frac{H}{\beta} - \frac{\eta}{p d^{-\alpha}}\right)^{-\delta}\right]^{-1}.
\end{aligned} \tag{6}$$

Channel inversion. *Consider the same network with $P_i = \frac{p}{\mathbb{E}[H^{-1}]} H_{ii}^{-1}$ for all i. Then the following expressions give tight bounds on the outage probability and transmission attempt intensity for λ, ϵ small:*

$$\begin{aligned}
q^{\mathrm{ci}}(\lambda) &\geq q_l^{\mathrm{ci}}(\lambda) = 1 - \exp\left\{-\lambda\pi d^2 \mathbb{E}[H^\delta]\mathbb{E}[H^{-\delta}]\left(\frac{1}{\beta} - \frac{\eta\mathbb{E}[H^{-1}]}{p d^{-\alpha}}\right)^{-\delta}\right\} \\
\lambda^{\mathrm{ci}}(\epsilon) &\leq \lambda_u^{\mathrm{ci}}(\epsilon) = -\log(1-\epsilon)\frac{1}{\pi d^2}\frac{1}{\mathbb{E}[H^\delta]\mathbb{E}[H^{-\delta}]}\left(\frac{1}{\beta} - \frac{\eta\mathbb{E}[H^{-1}]}{p d^{-\alpha}}\right)^{\delta}.
\end{aligned} \tag{7}$$

Note that *cp* denotes constant power, while *ci* denotes channel inversion. The constant power outage probability approximation $q_l^{\mathrm{cp}}(\lambda) \approx \tilde{q}_l^{\mathrm{cp}}(\lambda)$ holds because e^{-x} is nearly linear for small x. The constant power transmission attempt intensity approximation is obtained by solving $\tilde{q}_l^{\mathrm{cp}}(\lambda) = \epsilon$ for λ. Similarly, the channel inversion transmission attempt intensity upper bound is obtained by solving $q_l^{\mathrm{ci}}(\lambda) = \epsilon$ for λ. The validity of the approximations is evaluated in the numerical and simulation results in Section IV. The following corollary illustrates the simplification of the above results when the noise may be ignored.

Corollary 2: When $\eta = 0$ the expressions in Theorem 2 simplify to:

$$\begin{aligned}
q^{\mathrm{cp}}(\lambda) &\geq q_l^{\mathrm{cp}}(\lambda) = 1 - \mathbb{E}\left[\exp\left\{-\lambda\pi d^2\beta^\delta \mathbb{E}\left[H^\delta\right] H^{-\delta}\right\}\right] \\
&\approx \tilde{q}_l^{\mathrm{cp}}(\lambda) = 1 - \exp\left\{-\lambda\pi d^2\beta^\delta \mathbb{E}\left[H^\delta\right]\mathbb{E}\left[H^{-\delta}\right]\right\}, \\
q^{\mathrm{ci}}(\lambda) &\geq q_l^{\mathrm{ci}}(\lambda) = 1 - \exp\left\{-\lambda\pi d^2\beta^\delta \mathbb{E}\left[H^\delta\right]\mathbb{E}\left[H^{-\delta}\right]\right\}, \\
\lambda^{\mathrm{cp}}(\epsilon) &\approx \tilde{\lambda}^{\mathrm{cp}}(\epsilon) = -\log(1-\epsilon)\frac{1}{\pi d^2\beta^\delta}\frac{1}{\mathbb{E}\left[H^\delta\right]\mathbb{E}\left[H^{-\delta}\right]}, \\
\lambda^{\mathrm{ci}}(\epsilon) &\leq \lambda_u^{\mathrm{ci}}(\epsilon) = -\log(1-\epsilon)\frac{1}{\pi d^2\beta^\delta}\frac{1}{\mathbb{E}\left[H^\delta\right]\mathbb{E}\left[H^{-\delta}\right]}.
\end{aligned} \tag{8}$$

Note that in the absence of noise the constant power outage probability approximation equals the channel inversion outage probability lower bound, $\tilde{q}_l^{\mathrm{cp}}(\lambda) = q_l^{\mathrm{ci}}(\lambda)$. Moreover, the constant power transmission attempt intensity approximation equals the channel inversion transmission attempt intensity upper bound, $\tilde{\lambda}^{\mathrm{cp}} = \lambda_u^{\mathrm{ci}}(\epsilon)$. Comparing $\tilde{\lambda}^{\mathrm{cp}} = \lambda_u^{\mathrm{ci}}(\epsilon)$ in (8) with $\lambda_u^{\mathrm{pl}}(\epsilon)$ in (5) it is evident that the impact of fading on the transmission capacity is measured by the loss factor, $L^{\mathrm{cp}} = L^{\mathrm{ci}}$, defined as

$$L^{\mathrm{cp}} = L^{\mathrm{ci}} = \frac{1}{\mathbb{E}\left[H^\delta\right]\mathbb{E}\left[H^{-\delta}\right]} < 1. \tag{9}$$

The inequality is obtained by applying Jensen's inequality to the convex function $1/x$ and the random variable H^δ. If constant power is used, the $\mathbb{E}[H^{-\delta}]$ term is due to fading of the desired signal while the $\mathbb{E}[H^\delta]$ term is due to fading of the interfering links. Fading of the interfering signal has a positive effect while fading of the desired signal has a negative effect. If channel inversion is performed the $\mathbb{E}[H^{-\delta}]$ term is due to each interfering transmitter using power proportional to H_{ii}^{-1}. When the path loss exponent, α, is close to 2 then $\delta = 2/\alpha$ is close to one, so the term $\mathbb{E}[H^{-\delta}]$ is nearly equal to the expectation of the inverse of the fading, which can be extremely large for severe fading distributions such as Rayleigh. As a less severe example, $\alpha = 3$, the loss factor for Rayleigh fading is $L^{\mathrm{cp}} = L^{\mathrm{ci}} = 0.41$.

B. Transmission capacity under fractional power control

In this section we generalize the results of Theorem 2 by introducing fractional power control (fpc) with parameter $s \in [0,1]$. Under fpc the transmission power is set to $P_i = \frac{p}{\mathbb{E}[H^{-s}]} H_{ii}^{-s}$ for each i. The received power at receiver i is then $P_i H_{ii} d^{-\alpha} = \frac{p}{\mathbb{E}[H^{-s}]} H_{ii}^{1-s} d^{-\alpha}$, which depends upon i aside from $s=1$. The expected transmission power is p, ensuring a fair comparison with the results in Theorems 1 and 2. Note that constant power corresponds to $s=0$ and channel inversion corresponds to $s=1$. The following theorem gives good approximations on the outage probability and transmission attempt intensity under fpc.

Theorem 3: **Fractional power control.** *Consider a network where the SINR at the reference receiver is given by (3) with $P_i = \frac{p}{\mathbb{E}[H^{-s}]} H_{ii}^{-s}$ for all i, for some $s \in [0,1]$. Then the following expressions give good approximations of the outage probability and transmission attempt intensity for λ, ϵ small:*

$$\begin{aligned} q^{\mathrm{fpc}}(\lambda) &\geq q_l^{\mathrm{fpc}}(\lambda) = 1 - \mathbb{E}\left[\exp\left\{-\lambda\pi d^2 \mathbb{E}[H^{-s\delta}]\mathbb{E}[H^{\delta}]\left(\frac{H^{1-s}}{\beta} - \frac{\eta\mathbb{E}[H^{-s}]}{pd^{-\alpha}}\right)^{-\delta}\right\}\right] \\ &\approx \tilde{q}_l^{\mathrm{fpc}}(\lambda) = 1 - \exp\left\{-\lambda\pi\mathbb{E}[H^{-s\delta}]\mathbb{E}[H^{\delta}]\mathbb{E}\left[\left(\frac{H^{1-s}}{\beta} - \frac{\eta\mathbb{E}[H^{-s}]}{pd^{-\alpha}}\right)^{-\delta}\right]\right\}, \\ \lambda^{\mathrm{fpc}}(\epsilon) &\approx \tilde{\lambda}^{\mathrm{fpc}}(\epsilon) = -\log(1-\epsilon)\frac{1}{\pi d^2}\frac{1}{\mathbb{E}[H^{-s\delta}]\mathbb{E}[H^{\delta}]}\mathbb{E}\left[\left(\frac{H^{1-s}}{\beta} - \frac{\eta\mathbb{E}[H^{-s}]}{pd^{-\alpha}}\right)^{-\delta}\right]^{-1}. \end{aligned} \tag{10}$$

Proof: Under fpc, the SINR is given by:

$$\mathrm{SINR}_0 = \frac{P_0 H_0 d^{-\alpha}}{\sum_{i\in\Pi(\lambda)} P_i H_i X_i^{-\alpha} + \eta} = \frac{\frac{p}{\mathbb{E}[H^{-s}]} H_0^{1-s} d^{-\alpha}}{\sum_{i\in\Pi(\lambda)} \frac{p}{\mathbb{E}[H^{-s}]} H_{ii}^{-s} H_i X_i^{-\alpha} + \eta} \tag{11}$$

Therefore, the outage probability is given by:

$$q^{\mathrm{fpc}}(\lambda) = \mathbb{P}\left(\sum_{i\in\Pi(\lambda)} H_{ii}^{-s} H_i X_i^{-\alpha} > \frac{1}{\beta} H_0^{1-s} d^{-\alpha} - \frac{\eta\mathbb{E}[H^{-s}]}{p}\right). \tag{12}$$

As shown in [4], if $\Pi(\lambda) = \{(X_i, Z_i)\}$ is a homogeneous marked Poisson point process with points $\{X_i\}$ of intensity λ and iid marks $\{Z_i\}$ independent of the $\{X_i\}$, then

$$\mathbb{P}\left(\sum_{i\in\Pi(\lambda)} Z_i X_i^{-\alpha} > y\right) \geq 1 - \exp\left\{-\pi\lambda\mathbb{E}[Z^{\delta}] y^{-\delta}\right\}, \tag{13}$$

where the bound is asymptotically tight as $y \to \infty$. Conditioned on H_0 the RHS in (12) is a constant, hence we define:

$$Z_i = H_{ii}^{-s} H_i, \quad y = \frac{1}{\beta} H_0^{1-s} d^{-\alpha} - \frac{\eta\mathbb{E}[H^{-s}]}{p}. \tag{14}$$

Applying the above result yields:

$$q^{\mathrm{fpc}}(\lambda) \geq q_l^{\mathrm{fpc}}(\lambda) = 1 - \mathbb{E}\left[\exp\left\{-\pi\lambda\mathbb{E}\left[\left(H_{ii}^{-s} H_i\right)^{\delta}\right]\left(\frac{1}{\beta} H_0^{1-s} d^{-\alpha} - \frac{\eta\mathbb{E}[H^{-s}]}{p}\right)^{-\delta}\right\}\right]. \tag{15}$$

Recall that H_{ii} and $H_i = H_{i0}$ are assumed to be independent random variables. ■

As with Theorem 2, the approximation $q_l^{\mathrm{fpc}}(\lambda) \approx \tilde{q}_l^{\mathrm{fpc}}(\lambda)$ holds because e^{-x} is nearly linear for small x. The fpc transmission attempt intensity approximation, $\tilde{\lambda}^{\mathrm{fpc}}(\epsilon)$, is obtained by solving $\tilde{q}_l^{\mathrm{fpc}}(\lambda) = \epsilon$ for λ. The following corollary illustrates the simplification of the above results when the noise may be ignored.

Corollary 3: When $\eta = 0$ the expressions in Theorem 3 simplify to:

$$\begin{aligned} q^{\mathrm{fpc}}(\lambda) &\geq q_l^{\mathrm{fpc}}(\lambda) = 1 - \mathbb{E}\left[\exp\left\{-\lambda\pi d^2\beta^{\delta}\mathbb{E}\left[H^{\delta}\right]\mathbb{E}\left[H^{-s\delta}\right]H^{-(1-s)\delta}\right\}\right] \\ &\approx \tilde{q}_l^{\mathrm{fpc}}(\lambda) = 1 - \exp\left\{-\lambda\pi d^2\beta^{\delta}\mathbb{E}\left[H^{\delta}\right]\mathbb{E}\left[H^{-s\delta}\right]\mathbb{E}\left[H^{-(1-s)\delta}\right]\right\}, \\ \lambda^{\mathrm{fpc}}(\epsilon) &\approx \tilde{\lambda}^{\mathrm{fpc}}(\epsilon) = -\log(1-\epsilon)\frac{1}{\pi d^2\beta^{\delta}}\frac{1}{\mathbb{E}\left[H^{\delta}\right]\mathbb{E}\left[H^{-s\delta}\right]\mathbb{E}\left[H^{-(1-s)\delta}\right]}. \end{aligned} \tag{16}$$

The loss factor for fpc, L^{fpc}, is the reduction in the transmission capacity approximation relative to the pure pathloss case:

$$L^{\text{fpc}}(s) = \frac{1}{\mathbb{E}\left[H^{\delta}\right]\mathbb{E}\left[H^{-s\delta}\right]\mathbb{E}\left[H^{-(1-s)\delta}\right]}. \tag{17}$$

Clearly, the loss factor L^{fpc} for fpc depends on the design choice of the exponent s.

C. Optimal Fractional Power Control

Fractional power control represents a balance between the extremes of no power control and channel inversion. The mathematical effect of fractional power control is to replace the $\mathbb{E}[H^{-\delta}]$ term with $\mathbb{E}[H^{-s\delta}]\mathbb{E}[H^{-(1-s)\delta}]$. This is because the signal fading is *softened* by the power control exponent $-s$ so that it results in a leading term of $H^{-(1-s)}$ (rather than H^{-1}) in the numerator of the SINR expression, and ultimately to the $\mathbb{E}[H^{-(1-s)\delta}]$ term. The interference power is also softened by the fractional power control and leads to the $\mathbb{E}[H^{-s\delta}]$ term. The key question of course lies in determining the optimal power control exponent. It is given by the following theorem.

Theorem 4: In the absence of noise ($\eta = 0$), the fractional power control outage probability approximation, $\tilde{q}_l^{\text{fpc}}(\lambda)$, is minimized for $s = \frac{1}{2}$. Hence, the fractional power control transmission attempt intensity approximation, $\tilde{\lambda}^{\text{fpc}}(\epsilon)$ is also maximized for $s = \frac{1}{2}$.

Proof: To prove the theorem we must show that the quantity $\mathbb{E}\left[H^{-s\delta}\right]\mathbb{E}\left[H^{-(1-s)\delta}\right]$ is minimized at $s = \frac{1}{2}$. To do this, we apply the following general result with $X = H^{\delta}$, which we prove in the Appendix. For any non-negative random variable X, the function

$$h(s) = \mathbb{E}\left[X^{-s}\right]\mathbb{E}\left[X^{s-1}\right], \tag{18}$$

is convex in s for $s \in \mathbb{R}$ with a unique minimum at $s = \frac{1}{2}$. Since $\tilde{\lambda}^{\text{fpc}}(\epsilon)$ is inversely proportional to $\mathbb{E}\left[H^{-s\delta}\right]\mathbb{E}\left[H^{-(1-s)\delta}\right]$ with no other dependence on s, then clearly $s = \frac{1}{2}$ maximizes the transmission capacity approximation. ■

The theorem shows that transmission density is maximized, or equivalently, outage probability is minimized, by balancing the positive and negative effects of power control, which are reduction of signal fading and increasing interference, respectively. Using an exponent greater than $\frac{1}{2}$ over-compensates for signal fading and leads to interference levels that are too high, while using an exponent smaller than $\frac{1}{2}$ leads to small interference levels but an under-compensation for signal fading. Note that because the key expression $\mathbb{E}\left[H^{-s\delta}\right]\mathbb{E}\left[H^{-(1-s)\delta}\right]$ is convex, the loss relative to using $s = \frac{1}{2}$ increases monotonically both as $s \to 0$ and $s \to 1$.

One can certainly envision “fractional” power control schemes that go even further. For example, $s > 1$ corresponds to “super” channel inversion, in which bad channels take resources from good channels even more so than in normal channel inversion. Not surprisingly, this is not a wise policy. Less obviously, $s < 0$ corresponds to what is sometimes called “greedy” optimization, in which good channels are given more resources at the further expense of poor channels. Waterfilling is an example of a greedy optimization procedure. But, since $\mathbb{E}\left[H^{-s\delta}\right]\mathbb{E}\left[H^{-(1-s)\delta}\right]$ monotonically increases as s decreases, it is clear that greedy power allocations of any type are worse than even constant transmit power.

The fpc transmission attempt intensity approximation at the optimal exponent $s = \frac{1}{2}$ is given by

$$\tilde{\lambda}^{\text{fpc}}(\epsilon) \approx -\log(1-\epsilon)\frac{1}{\pi d^2 \beta^{\delta}}\frac{1}{\mathbb{E}\left[H^{\delta}\right]\left(\mathbb{E}\left[H^{-\frac{\delta}{2}}\right]\right)^2}. \tag{19}$$

The numerical results given in the proceeding pages show that this optimized density can be significantly higher than that achieved without any power control or with channel inversion. However, it should be noted that fading has a deleterious effect relative to no fading even if the optimal exponent is used. To see this, note that $x^{-\frac{1}{2}}$ is a convex function and therefore Jensen's yields $\mathbb{E}[X^{-\frac{1}{2}}] \geq (\mathbb{E}[X])^{-\frac{1}{2}}$ for any non-negative random variable X. Applying this to $X = H^{\delta}$ we get $\left(\mathbb{E}\left[H^{-\frac{\delta}{2}}\right]\right)^2 \geq \left(\mathbb{E}[H^{\delta}]\right)^{-1}$, which implies

$$L^{\text{fpc}}(1/2) = \frac{1}{\mathbb{E}\left[H^{\delta}\right]\left(\mathbb{E}\left[H^{-\frac{\delta}{2}}\right]\right)^2} \leq 1.$$

Therefore, fractional PC cannot fully overcome fading, but it is definitely a better power control policy than constant power transmission or traditional power control (channel inversion).

IV. Numerical Results

The benefit of fractional power control is well illustrated by examining performance in Rayleigh fading, in which case H is exponentially distributed and the moment generating function is therefore

$$\mathbb{E}[H^t] = \Gamma(1+t), \tag{20}$$

where $\Gamma(\cdot)$ is the standard gamma function. If fractional power control is used, the effect of fading is

$$L^{\rm fpc} = \frac{1}{\mathbb{E}\left[H^{\delta}\right]\mathbb{E}\left[H^{-s\delta}\right]\mathbb{E}\left[H^{(1-s)\delta}\right]} = \frac{1}{\Gamma(1+\delta)\cdot\Gamma(1-s\delta)\cdot\Gamma(1-(1-s)\delta)} \tag{21}$$

In Figure 1 this loss factor (L) is plotted as a function of s for path loss exponents $\alpha = \{2.1, 3, 4\}$. Notice that for each value of α the maximum takes place at $s = \frac{1}{2}$, and that the cost of not using fractional power control is highest for small path loss exponents because $\Gamma(1+x)$ goes to infinity quite steeply as $x \to -1$.

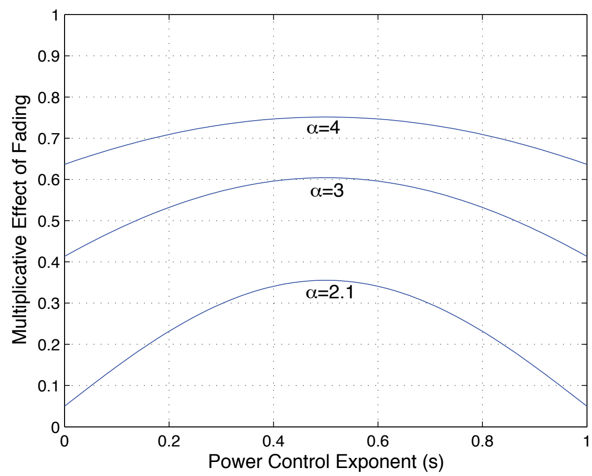

Fig. 1. The loss factor L vs. s. Note that $L^{\rm cp}$ and $L^{\rm ci}$ are the left edge and right edge of the plot, respectively.

In Figure 2 the loss factor L is again plotted against the path loss exponent α for $s = \frac{1}{2}$ and for no power control/channel inversion (i.e. $s = 0$ or $s = 1$). If no power control or channel inversion is used ($s = 0$ or $s = 1$) in a network with a path loss exponent near two, Rayleigh fading almost completely zeroes the transmission capacity. This is because $L \to 0$ as $s \to 0$ or $s \to 1$, coupled with the fact that the mean interference is infinite in a Poisson field of interferers with $\alpha < 2$ [6]. However, if fractional PC with $s = 1/2$ is used, it is feasible to operate in Rayleigh fading because $L^{\rm fpc} > \frac{1}{\gamma(0.5)^2} = 0.3183$ for all path loss exponents greater than 2.

Although fractional power control is quite powerful with respect to no/full power control, fading still has a significant effect on transmission capacity because the multiplicative loss factor L is still non-negligibly far from one. For example, for $\alpha = 3$ Rayleigh fading still reduces capacity by a factor of 0.6 with fractional PC (rather than by 0.4 with no PC). In Figure 3 the multiplicative savings of optimal fractional power control relative to no power control/channel inversion, which is the quantity

$$\frac{L^{\rm fpc}}{L^{\rm cp}} = \frac{\Gamma(1-\delta)}{\left(\Gamma(1-\frac{1}{2}\delta)\right)^2}, \tag{22}$$

is shown. For path loss exponents near 2, there is a very substantial performance increase. This advantage is reduced for larger values of α, but is still quite significant.

The above numerical results are in terms of the approximation to transmission capacity. Of course, it is also necessary to show how accurate this approximation is. Figure 4 presents simulation results for the outage probability, $q(\lambda)$, versus the fpc exponent, s, for $\lambda = 0.02$ in Rayleigh fading (H exponential). The simulation results are Monte Carlo averages. The simulation results assume

$$\alpha = 4 \quad \beta = 1 \quad d = 1 \quad p = 1 \quad \eta = 10^{-6}. \tag{23}$$

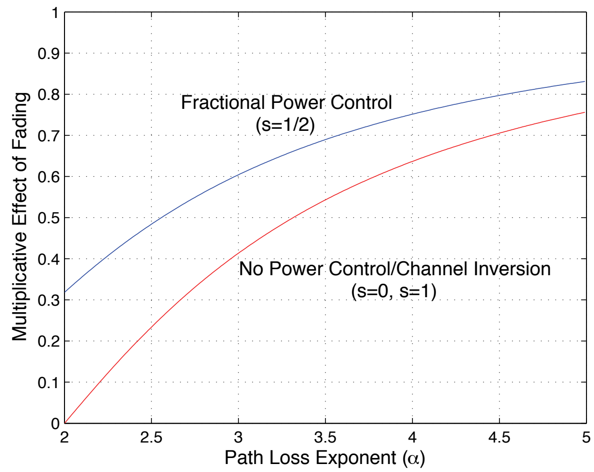

Fig. 2. Fading loss factor L for fractional power control, channel inversion, and constant power transmission.

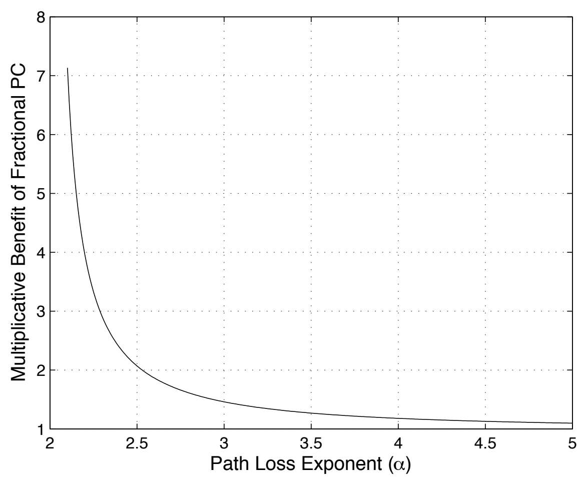

Fig. 3. The ratio of transmission capacity for fractional power control ($s = \frac{1}{2}$) relative to no power control or channel inversion. The gain from FPC becomes quite significant as $\alpha \to 2$.

The no noise plot (i.e., SIR) corresponds to $\eta = 0$, while the curve with noise was computed for $\eta = 10^{-2}$ (i.e., a received SNR of 20 dB in the absence of any interference). The figure illustrates a close match between the simulated outage probability and the numerical approximation in (16), and note that the optimal exponent is 0.5 with or without noise. It is interesting to note that the outage probability with noise diverges for s near one; this is because the power cost of Rayleigh fading goes to infinity as s approaches one.

V. Areas for Future Study

Given the historically very high level of interest in the subject of power control for wireless systems, this new paradigm for power control opens many new questions. We document some that have occurred to us here, but note that this list is certainly not exhaustive.

What is the effect of scheduling on FPC? If scheduling is used, then how should power levels between a transmitter and receiver be set? Will $s = \frac{1}{2}$ still be optimal? Will the gain be increased or reduced? We conjecture that the gain from FPC will be smaller but non-zero for most any sensible scheduling policy, as the effect of interference inversion is softened.

How does FPC perform in cellular systems?. Cellular systems in this case are harder to analyze than ad hoc networks, because the base stations (receivers) are located on a regular grid and thus the tractability of the spatial

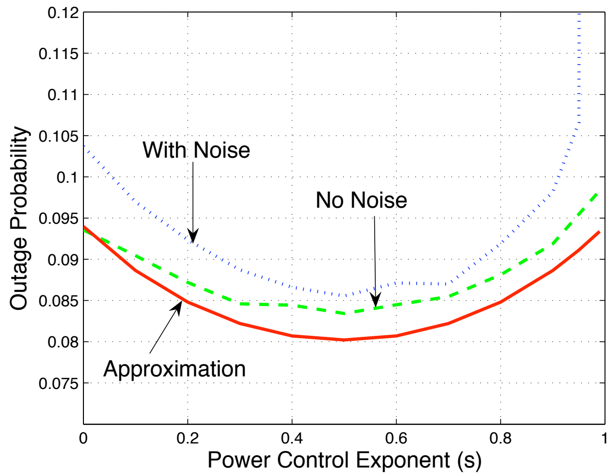

Fig. 4. Outage probability vs. power control exponent

Poisson model cannot be exploited. On the other hand, FPC may be even more helpful in centralized systems. Note that some numerical results for cellular systems are given in reference [2], but no analysis is provided.

VI. Conclusions

This paper has applied fractional power control as a general approach to pairwise power control in ad hoc networks. We have shown that under some assumptions, the optimum power control exponent is $s^* = \frac{1}{2}$, in contrast to constant transmit power ($s = 0$) or channel inversion ($s = 1$). This implies that there is an optimal balance between compensating for fades in the desired signal and amplifying interference. We saw that a gain on the order of 50% or larger (relative to no power control) might be typical for fractional power control in a typical wireless channel. The gains are larger for channels with low attenuation, and smaller for channels with high attenuation.

Appendix

We prove that for any non-negative random variable X, the function

$$h(s) = \mathbb{E}\left[X^{-s}\right]\mathbb{E}\left[X^{s-1}\right], \tag{24}$$

is convex in s for $s \in \mathbb{R}$ with a unique minimum at $s = \frac{1}{2}$. Define the functions

$$f(s) = \mathbb{E}\left[X^{-s}\right], \quad g(s) = \mathbb{E}\left[X^{s-1}\right], \tag{25}$$

so that $h(s) = f(s)g(s)$. Recall that a function h is said to be *log-convex* if $\log h$ is convex. Two relevant properties of log-convex functions are: $i)$ if h is log-convex then h is convex (although the converse need not hold), and $ii)$ the product of two log-convex functions is log-convex. Thus the theorem will be proved if we show that f, g are log-convex. The functions f, g are easily shown to be convex by the fact that $f''(s), g''(s)$ are non-negative for all s:

$$\begin{aligned} f'(s) &= -\mathbb{E}\left[X^{-s}\log X\right], & f''(s) &= \mathbb{E}\left[X^{-s}(\log X)^2\right] \\ g'(s) &= \mathbb{E}\left[X^{s-1}\log X\right], & g''(s) &= \mathbb{E}\left[X^{s-1}(\log X)^2\right], \end{aligned} \tag{26}$$

Define $F(s) = \log f(s)$, $G(s) = \log g(s)$, and $H(s) = \log h(s)$. Then:

$$\begin{aligned} F'(s) &= \frac{-\mathbb{E}[X^{-s}\log X]}{\mathbb{E}[X^{-s}]}, & F''(s) &= \frac{\mathbb{E}\left[X^{-s}\right]\mathbb{E}\left[X^{-s}(\log X)^2\right] - \left(\mathbb{E}\left[X^{-s}\log X\right]\right)^2}{\left(\mathbb{E}\left[X^{-s}\right]\right)^2} \\ G'(s) &= \frac{-\mathbb{E}[X^{s-1}\log X]}{\mathbb{E}[X^{s-1}]}, & G''(s) &= \frac{\mathbb{E}[X^{s-1}]\mathbb{E}[X^{s-1}(\log X)^2] - \left(\mathbb{E}\left[X^{s-1}\log X\right]\right)^2}{\left(\mathbb{E}\left[X^{s-1}\right]\right)^2}. \end{aligned} \tag{27}$$

We employ the Cauchy-Schwarz inequality in the form

$$\left(\mathbb{E}\left[a(X)b(X)\right]\right)^2 \leq \mathbb{E}\left[a(X)^2\right]\mathbb{E}\left[b(X)^2\right], \tag{28}$$

for arbitrary functions a, b of a random variable X.

$$\begin{aligned} F^{''}(s) &= \frac{\mathbb{E}\left[\left(X^{-\frac{s}{2}}\right)^2\right]\mathbb{E}\left[\left(X^{-\frac{s}{2}}\log X\right)^2\right] - \left(\mathbb{E}\left[X^{-s}\log X\right]\right)^2}{\left(\mathbb{E}\left[X^{-s}\right]\right)^2} \\ &\geq \frac{\mathbb{E}\left[X^{-\frac{s}{2}}X^{-\frac{s}{2}}\log X\right]^2 - \left(\mathbb{E}\left[X^{-s}\log X\right]\right)^2}{\left(\mathbb{E}\left[X^{-s}\right]\right)^2} = 0 \\ G^{''}(s) &= \frac{\mathbb{E}\left[\left(X^{\frac{s-1}{2}}\right)^2\right]\mathbb{E}\left[\left(X^{\frac{s-1}{2}}\log X\right)^2\right] - \left(\mathbb{E}\left[X^{s-1}\log X\right]\right)^2}{\left(\mathbb{E}\left[X^{s-1}\right]\right)^2} \\ &\geq \frac{\mathbb{E}\left[X^{\frac{s-1}{2}}X^{\frac{s-1}{2}}\log X\right]^2 - \left(\mathbb{E}\left[X^{s-1}\log X\right]\right)^2}{\left(\mathbb{E}\left[X^{s-1}\right]\right)^2} = 0 \end{aligned} \tag{29}$$

This establishes the log-convexity of f, g, and hence the log-convexity of h, and thus the convexity of h. The derivative of h is

$$h'(s) = \mathbb{E}\left[X^{-s}\right]\mathbb{E}\left[X^{s-1}\log X\right] - \mathbb{E}\left[X^{s-1}\right]\mathbb{E}\left[X^{-s}\log X\right]. \tag{30}$$

It can easily be seen that $s = \frac{1}{2}$ is the unique maximizer satisfying $h'(s) = 0$.

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