Capacity of the Cognitive Tracking Channel

Sudhir Srinivasa and Syed Ali Jafar
Electrical Engineering and Computer Science
University of California Irvine, Irvine, CA 92697-2625
Email: *sudhirs@uci.edu, syed@ece.uci.edu*

Nihar Jindal
Department of Electrical and Computer Engineering
University of Minnesota, Minneapolis, MN 55455
Email: *nihar@umn.edu*

***Abstract*— Abstract**

I. Introduction

Cognitive radios [1]–[4] are encouraging solutions to improve the utilization of the radio spectrum. The main idea in cognitive radio is to periodically monitor the radio spectrum, intelligently detect occupancy in the different parts (channels) of the spectrum and then opportunistically communicate over unused channels (spectrum holes) with minimal interference to the active licensed (primary) users. Cognitive radio, however, is still an emerging technology and faces a number of challenges in how the radio learns and adapts to the local spectral activity at each end of the cognitive link.

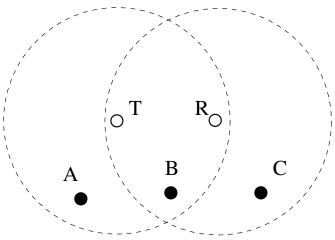

Fig. 1: Different Perspectives on Local Spectral Activity at Cognitive Radio Transmitter T and Receiver R

Due to the physical separation of the secondary[1] transmitter and receiver, the spectral holes sensed by the transmitter may not be identical to those sensed by the receiver. Consider the conceptual depiction of a cognitive radio link shown in Figure 1. The white nodes marked T and R represent the cognitive radio transmitter and receiver respectively while the black nodes marked A, B and C are primary users occupying different parts of the spectrum. The dotted regions around the cognitive radio transmitter and receiver are the boundaries of the respective *sensing regions* - spectral activity can only be sensed within these regions. Because of the different sensing regions at the secondary transmitter and the receiver, the communication opportunities detected at the transmitter T and receiver R are in general correlated but may not be identical [5]. As the primary user activity changes with time, such opportunities have to be re-evaluated periodically. We use the term *distributed* to indicate the different views of local spectral activity at the cognitive transmitter T and receiver R. The term 'dynamic' refers to the time varying nature of the problem.

[1]Opportunistic access is only used by the cognitive (secondary) users in the system.

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Once the detection of the spectral holes is complete, the secondary transmitter picks one of the holes (if any) and exploits it for secondary communication. Similarly at the cognitive receiver, one of the spectral segments identified to be locally free is chosen to be monitored for secondary transmissions. Due to the distributed nature of the cognitive link, any communication scheme between the secondary transmitter and receiver is faced with two important issues:

1) **Matching problem:** Are the spectral holes chosen for communication by the transmitter and the receiver matched ? In other words, is the spectral segment picked by the secondary transmitter the same as that chosen to be monitored at the cognitive receiver ?
2) **Tracking problem:** If the transmitter and the receiver are indeed matched to the same spectral segment, does the cognitive receiver know that they are matched ?

These issues necessitate a handshake between the transmitter and the receiver before the beginning of communication. Such a handshake is associated with two overheads - one on the forward link from the transmitter to the receiver and the other, on the reverse link from the receiver to the transmitter. Notice that the two overheads are closely linked to the matching problem and the tracking problem.

An obvious solution to eliminate both the overheads is to have an 'offline' scheme where a pre-determined sequence of frequency bands is known to both the secondary transmitter and receiver. During any time slot n, the secondary transmitter monitors the n^{th} frequency band in the sequence, f_n and uses it for communication if it is primary user free. The receiver is always matched with the transmitter because it monitors the same frequency segment f_n for secondary transmissions. The drawback of the offline scheme is that it does not take advantage of the real-time 'online' information obtained about the channel at the cognitive receiver. For example, consider a scenario where the primary users' transmissions are long and infrequent, i.e. the primary users' occupancy is characterized by extended periods of activity followed by long periods where the primary user is inactive. Channels (frequency bands) that are free in a particular time slot are therefore more likely to be unoccupied even in the next time slot. Since the offline scheme does not allow the secondary transmitter to stay in the same channel for more than one time slot, frequency bands that are free are not reused. In this light, we note that it may be possible to use occupancy information of the past to predict current

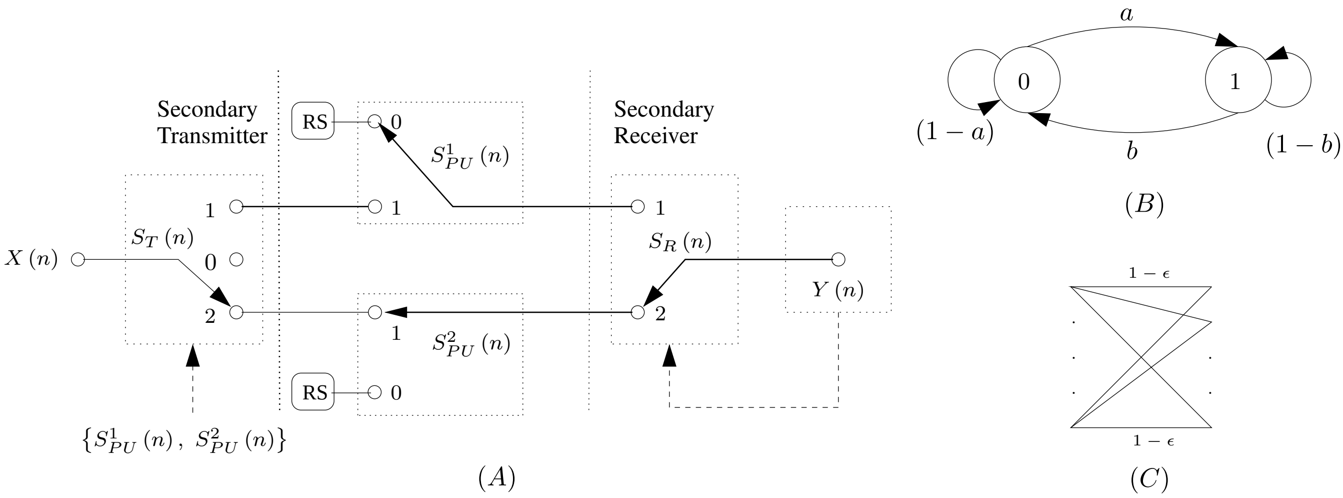

Fig. 2: Part (A) illustrates the system model. RS denotes a random source. At the time instant considered, we have $S_{PU}^1(n) = 0$, $S_{PU}^2(n) = 1$, $S_T(n) = 2$ and $S_R(n) = 2$. Part (B) shows the Markov chain for the primary user occupancies $S_{PU}^1(n)$ and $S_{PU}^2(n)$ in the two channels. Part (C) is the Q-ary symmetric channel.

occupancies, identify more communication opportunities and consequently perform better despite the overheads involved.

In order to evaluate the cost of the overhead information and to determine the benefits of these overheads to the cognitive user, a capacity perspective would be immensely helpful. Our goal in this paper is to characterize the fundamental limitations on the capacity of the cognitive user in a distributed environment. We begin with the system model in Section II.

II. System and Channel Model

Our system model is defined on a spectrum pool consisting of L channels assigned for use to different primary users. For simplicity of exposition, we will only deal with $L = 2$ channels in this work - analysis of cases where $L \geq 2$ will be postponed to [6]. We consider a secondary (cognitive) transmitter-receiver pair trying to communicate with one another using the two channels as shown in Figure 2 (A).

1) Channel Availability Model: The primary user occupancies on the two channels at time n are collected in the binary random processes $S_{PU}^1(n) \in \{0,\ 1\}$ and $S_{PU}^2(n) \in \{0,\ 1\}$. A value of $S_{PU}^l(n) = 0$ indicates that a primary user is actively transmitting on channel l at time n. Similarly, a value of $S_{PU}^l(n) = 1$ implies that channel l is free for secondary transmissions. We model the occupancy processes $S_{PU}^1(n)$ and $S_{PU}^2(n)$ with independent and identical Markov chains as shown in Figure 2 (B).

2) Cognitive Transmitter:: The secondary transmitter monitors the two channels every time slot to determine whether or not it is in use by the primary radios. When one or more of the two channels is/are deemed temporally unoccupied, it chooses *one* of the available channels for secondary transmissions. If neither of the two channels is detected to be free, the secondary transmitter does not transmit and goes to the idle state. Therefore in any time slot the cognitive transmitter is in one of three states - it is idle (no transmission), transmits on channel 1 or transmits on channel 2. We capture this state information in the variable $S_T(n) \in \{0,\ 1,\ 2\}$.

3) Input Alphabet and Probability of error: The input alphabet used by the cognitive transmitter is assumed to be a Q-ary constellation with equiprobable symbols. We further assume that the two channels are Q-ary symmetric with a symbol error probability of ϵ as shown in Figure 2 (C). The discrete symmetric channels can be thought of as arising due to hard decision decoding at the secondary receiver. ϵ is a function of the constellation size Q and the constellation power used at the secondary/primary transmitters.

4) Cognitive Receiver: The secondary receiver, at any given time, is only able to scan a single channel for secondary transmissions. It picks the channel which it considers will be used by the secondary transmitter in the next time slot and then listens to the corresponding frequency for secondary transmissions. We represent the receiver state at time n, i.e. the channel the receiver chooses to monitor during time slot n, by the variable $S_R(n) \in \{1,\ 2\}$.

For simplicity of analysis, we make the optimistic assumption that during each time slot, the secondary transmitter and receiver have accurate information of the presence/absence of primary users in both the channels in their local vicinities, i.e., no errors are made in the detection of spectral holes. We analyze and discuss the effect of missed detection and false alarm [7] on the capacity in [6].

We denote the symbol transmitted from the cognitive transmitter (if any) at time n by $X(n)$ and the corresponding signal received by $Y(n)$. When the transmitter and receiver states are matched ($S_T(n) = S_R(n)$), $Y(n)$ and $X(n)$ are related through the probability of error distribution dictated by the Q-ary symmetric channel (QSC) model. On the other hand, when $S_T(n) \neq S_R(n)$ (includes $S_T(n) = 0$), the cognitive receiver only sees random signals (if the primary user is transmitting) or noise - both of which we assume are demodulated to one of the Q constellation symbols with equal likelihood.

The transmitter only has knowledge of its state $S_T(n)$ and similarly the receiver only about its own state $S_R(n)$.

Communication during time slot n is feasible only if the secondary transmitter state matches that of the secondary receiver, i.e., $S_T(n) = S_R(n)$. The receiver, therefore, has to track the transmitter state before it can monitor the corresponding channel and then decode the received signals.

III. Capacity Perspective

In a single user point-to-point scenario with different causal side information at the transmitter and the receiver, the system capacity is the solution to the following optimization problem [8]:

$$C = \max_{p_u(U),\ X=f(U;\ S_T)} \mathcal{I}(U;\ Y, S_R), \tag{1}$$

where X and Y represent the input and the output, S_T and S_R denote the side information at the transmitter and the receiver and U is an auxiliary random variable independent of S_T. The basic premise of the capacity definition of equation (1) is that the channel definition $\text{Prob}[Y,\ S_R | X,\ S_T]$ is known.

One might be tempted to use equation (1) to determine the capacity of the cognitive link in the system model of Section II. However, it should be noted that the channel definition $\text{Prob}[Y(n),\ S_R(n) | X(n),\ S_T(n)]$ in the model is not complete because it critically depends on the following:

- **Transmitter strategy:** The strategy used at the transmitter to choose the channel for transmission when more than one spectral hole is detected.
- **Receiver strategy:** The strategy used at the receiver to track the transmitter state $S_T(n)$.

Consequently equation (1) *cannot* be used for our purposes[2].

In the sequel, we assume that the transmitter strategy is known. Note, however, that this assumption does not simplify our problem - since the receiver strategy is not known, to apply equation (1) warrants a maximization of the mutual information over all possible receiver strategies. This is mathematically intractable and consequently we discount bounding the system capacity by the conventional mutual information optimization. We approach the problem instead through the notion of matching probability which, as we show, can be used to bound the system capacity given only the transmitter strategy.

We define the matching probability α as the average fraction of time the secondary transmitter and receiver are, as the name implies, matched to the same channel, i.e.,

$$\alpha = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} \mathbf{I}[S_T(n) = S_R(n)]}{N}, \tag{2}$$

where N is the codeword duration and $\mathbf{I}[\cdot]$ denotes the indicator function. If both the transmitter and the receiver strategies are known, the computation of the matching probability α is straightforward. However, as we pointed out earlier, we only assume that the transmitter strategy is given. Using the concept of matching probability, we now discuss how upper and lower bounds on the system capacity (for the given transmit strategy) can be derived.

A. Upper Bounds

For every set of transmitter and receiver strategies, there exists a corresponding matching probability α. It is easy to see that the capacity (for the given transmitter receiver strategies) cannot exceed $\alpha \cdot C_{QSC}(Q)$, where $C_{QSC}(Q)$ is the capacity of the Q-ary symmetric channel. Suppose we fix the transmitter strategy and determine an upperbound β on α such that $\alpha \le \beta$ for all possible receiver strategies. Let C_{S_T,S_R} be the capacity of the system of Section II for a given transmitter strategy. Clearly β can be used to upperbound this capacity because

$$\alpha \le \beta \Rightarrow C_{S_T,S_R} \le \alpha \cdot C_{QSC}(Q) \le \beta \cdot C_{QSC}(Q). \tag{3}$$

Using this idea, we will calculate different upperbounds on the capacity for an important transmit strategy in Section IV-C.

B. Lower Bounds

Any achievable scheme provides a lower bound on the capacity. A trivial lower bound would be the offline scheme discussed in Section I. Since the transmitter and the receiver are always matched, the receiver only needs to determine if the transmitter is active or idle. Using the genie bound [9], where the genie provides this information to the receiver, a lowerbound on the capacity of the cognitive link easily follows [6]:

$$C_{S_T,\ S_R} \ge \text{Prob}\left(S^1_{PU}(n) = 1\right) \cdot C_{QSC}(Q) - \mathcal{H}(G), \tag{4}$$

where $\text{Prob}\left(S^1_{PU}(n) = 1\right) = \text{Prob}\left(S^2_{PU}(n) = 1\right)$ is the steady state probability of the channel being free of primary users and $\mathcal{H}(G)$ is the entropy measure of the genie information provided to the receiver. This is also essentially the i.i.d scheme discussed in [5]. More sophisticated lower bounds can also be derived with achievable transmitter and receiver strategies. As an example, consider a scenario where the transmitter sends training symbols between the data symbols to help the receiver track the transmitter state. Capacity expressions similar to equation (4) can be derived even in this case. To summarize, given an achievable scheme the corresponding matching probability and the genie bound [9] can be used to derive a lower bound on the capacity.

In this light, the problem of bounding the capacity reduces to finding good upper bounds for the matching probability and tight achievable capacity inner bounds.

IV. Bounds on α for the Preference Strategy

In this section we consider the ‘preference strategy’ at the transmitter and present some outer bounds on the matching probability α. While the upperbound calculations are specific to the preference scheme, the technique employed is general and can be used for any transmitter strategy.

[2]If both the transmitter and receiver strategies are known, the channel characterization is complete and equation (1) can be used to obtain the capacity for the corresponding strategies.

A. Preference Transmitter Strategy

In this strategy, the secondary transmitter prefers one of the two channels over the other. Without any loss of generality, we assume that the preferred channel is channel $l = 1$. Under this policy, if both the channels are unoccupied, channel 1 is used. Table I lists the transmitter states depending on the primary user occupancies in the two channels.

$S_{PU}^{1}(n)$	$S_{PU}^{2}(n)$	$S_T(n)$
0	0	0
1	0	1
1	1	1
0	1	2

TABLE I: State choices for the preference policy.

The Markov chain for the transmitter state $S_T(n)$ can be easily derived from the Markov chains of the underlying channels $S_{PU}^{1}(n)$ and $S_{PU}^{2}(n)$ and is shown in Figure 3.

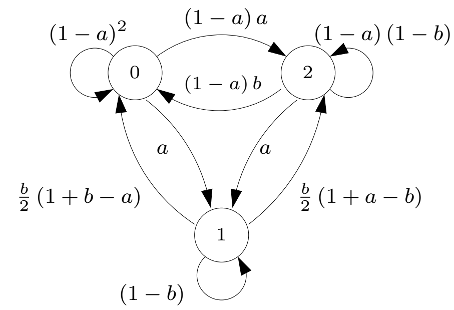

Fig. 3: Equivalent Markov chain for the transmitter state $S_T(n)$ for the preference strategy of Section IV-A.

B. Upper Bounds on α

1) Delayed Side Information at Receiver ($\alpha_{delayed}$): Delayed side information at the receiver refers to the case where the receiver has knowledge of the transmitter state in the previous time slot, i.e., at time n, the receiver chooses the channel to monitor in the next time slot, $S_R(n+1)$, using the knowledge of $S_T(n)$. The decision taken by the receiver is based on the maximum likelihood rule, i.e.,

$$k^* = \arg\max_{k\in\{1,\ 2\}} \text{Prob}\left[S_T(n) = k\,|S_T(n-1)\right] \quad (5)$$

The receiver decision rule therefore reduces to picking the non-zero state that has the highest transition probability from $S_T(n)$. Since the receiver has additional information about the previous state at the transmitter, it is easy to see that $\alpha_{\text{delayed}} \geq \alpha$.

When $a = b$, the preference policy yields steady state probabilities independent of a, given by $\left\{\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4}\right\}$ for the transmitter states $\{0,\ 1,\ 2\}$ respectively. In such a scenario it can be shown [6] that

$$\alpha_{\text{delayed}}(1) = \begin{cases} \frac{3-3\cdot a+a^2}{4} & a \leq \left(\frac{3-\sqrt{5}}{2}\right) \\ \frac{1}{2} & \left(\frac{3-\sqrt{5}}{2}\right) \leq a \leq \frac{2}{3} \\ \frac{3\cdot a}{4} & a \geq \frac{2}{3} \end{cases} \quad (6)$$

2) All $\mathbf{Y}$ Information ($\alpha_{\mathbf{Y}}$): Consider a scenario where the secondary receiver is able to monitor the received signals on *both* the channels. At the end of the time slot, however, it is required to output the estimate of the transmitter state in the next time slot, i.e., at the end of time slot n, the receiver has knowledge of $\mathcal{Y}(n) = \{\mathbf{Y}(0),\ \mathbf{Y}(1),\ \cdots,\ \mathbf{Y}(n)\}$ where $\mathbf{Y}(i) = [Y_1(i)\ \ Y_2(i)]$ is the received vector at time i. Based on $\mathcal{Y}(n)$ it has to determine $S_R(n+1)$. The optimal receiver strategy [6] in this case is to estimate the subsequent state $S_R(n+1)$ to be

$$k^* = \arg\max_{k\in\{1,\ 2\}} \text{Prob}\left[S_T(n+1) = k\,|\mathcal{Y}(n)\right] \quad (7)$$

The corresponding matching probability $\alpha_{\mathbf{Y}}$ is therefore an upperbound to α.

C. Achievable α Bounds

1) Offline Bound: For the offline scheme discussed earlier, the transmitter and the receiver are always matched. The best posible matching probability that can be achieved in this case is equal to the steady state probability of any of the identical channels being free.

For the $(a = b)$ case, $\alpha_{\text{offline}} = 0.5$ [6]. Since α_{delayed} is an upperbound on α, equation (6) shows that for all $\frac{3-\sqrt{5}}{2} \leq a \leq \frac{2}{3}$, the offline scheme is optimal in terms of achieving the best possible matching probability. The corresponding capacity bounds, however, differ in the genie information $\mathcal{H}(G)$.

2) Training Bound ($\alpha^*_{training}$): Consider the case where the secondary transmitter sends a known training symbol to the receiver once every N time slots (symbol periods). The training symbol is sent to help the secondary receiver track the state information of the secondary transmitter more reliably. Without any loss of generality, we assume that the training symbol sent is the first constellation symbol q_0, i.e., $X(k\cdot N) = q_0 \quad \forall\ k \in \{1,\ 2,\ \cdots\}$. Based on its state at $n = k\cdot N$, $S_R(k\cdot N)$ and the received signal $y(k\cdot N)$ the receiver decides either to continue monitoring the same channel or switch to the other channel. Between the training symbol time slots, the receiver is not allowed to change states and monitors the same channel chosen in the previous training symbol time slot. The resulting matching probability (not counting training symbols) can be used to lower bound the capacity.

Note that the matching probability $\alpha_{\text{training}}(N)$ is a function of the training delay N. if N is too small, due to the training overhead, $\alpha_{\text{training}}(N)$ is small. On the other hand when N is too large, since the receiver is not allowed to switch states between training symbols, the receiver cannot follow the transmitter state accurately. Consequently the resulting $\alpha_{\text{training}}(N)$ is low. There is therefore a tradeoff between the training delay and the matching probability. We define $\alpha^*_{\text{training}}$ as the maximum possible achievable matching probability and the corresponding optimal delay by N^*.

V. Simulation Results

In this section, we will provide numerical results on the capacity for the cognitive link of the system model of Section II. For our simulations we assume symmetric Markov chains for the primary user occupancy process, i.e., $a = b$.

Figure 4 plots the upper and lower bounds on the matching probability as a function of the loop probability of the primary user occupancy $\bar{a} = (1 - a)$ for $Q = 8$ and $\epsilon = 0.01$. The plot for α_{delayed} obtained from simulation follows that obtained from the analytical result of equation (6). We note here that the offline lower bound α_{offline} and α_{delayed} are defined to be independent of the constellation size Q and the error probability ϵ. Botht the plots therefore remain constant with increasing $\bar{a}$. The offline lower bound remains at the steady state probability Prob $\left[S^1_{PU}(n) = 1\right] = 0.5$. As $\bar{a}$ increases, the memory in the primary user occupancy process increases and the primary user states $S^l_{PU}(n)$ vary very slowly. Consequently it is easier to track the transmitter state in the training scheme discussed in Section IV-C.2. The corresponding matching probability $\alpha^*_{\text{training}}$ consequently increases. A similar argument holds for the increase of $\alpha_{\mathbf{Y}}$ with $\bar{a}$. Notice that the gap between α_{delayed} and $\alpha_{\mathbf{Y}}$ is very small - at the very low ($\epsilon = 0.01$) error probabilities involved, the performance of the corresponding schemes is almost the same.

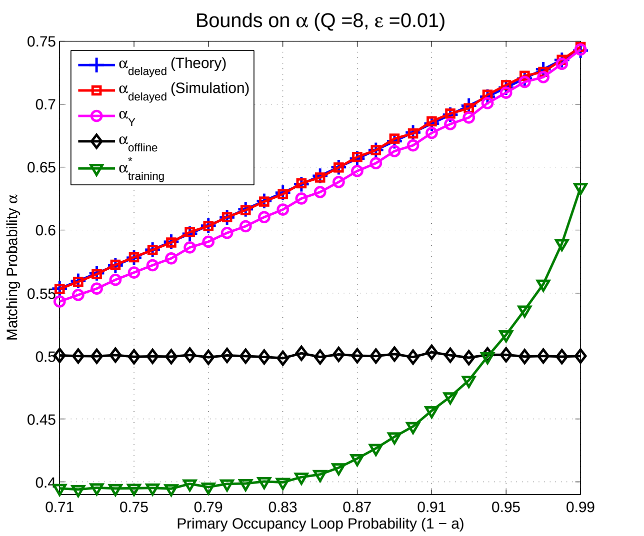

Fig. 4: Upper and achievable bounds on α with increasing $\bar{a}$ for $Q = 8$ and $\epsilon = 0.01$.

Similar plots for $Q = 2$ and $\epsilon = 0.2$ are shown in Figure 5. Since the probability of error $\epsilon = 0.2$ is higher, the matching probability seperation between α_{delayed} and $\alpha_{\mathbf{Y}}$ increases. Consider the upperbound plot $\alpha_{\mathbf{Y}}$ and the achievable lower-bound α_{offline}. The small gap indicates that for scenarios with high error probabilities ϵ, the offline scheme is fairly robust since if offers nearly the same matching probability as that possible in the upperbound (Equation (6) already shows that the offline scheme is α-optimal for $a \in \{0.3821, 0.666\}$.). The matching probability achieved by the training scheme suffers due to the fairly high error probability involved.

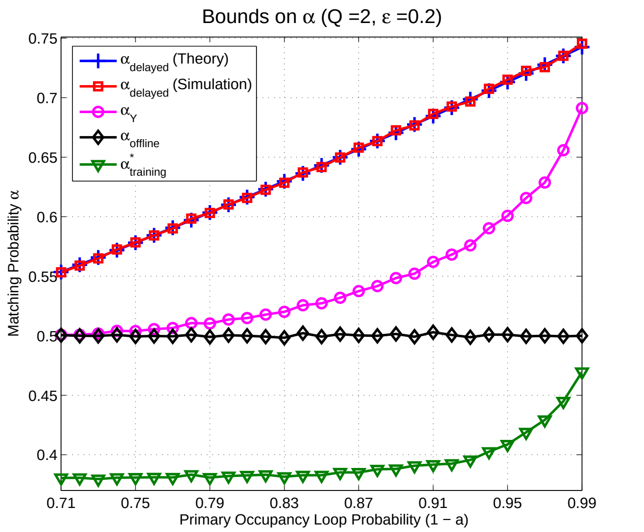

Fig. 5: Upper and achievable bounds on α with increasing $\bar{a}$ for $Q = 2$ and $\epsilon = 0.2$.

Other numerical results including capacity bounds for the above scenarios and matching probability/capacity plots for situations where $a \neq b$ are presented in [6]. For lack of space we also postpone the numerical analysis of the influence of the constellation power (which affects the resulting error probability) to [6].

VI. Conclusions

We explore the capacity of a cognitive radio system in a spectrum pool consisting of two channels with independent and identically distributed occupancy processes. The distributed channel information at either end of the cognitive link can be modeled with multi-state switches at the transmitter and receiver. The formulation of the problem precludes trying to determine the capacity by the conventional mutual information maximization. Using the probability of the receiver and transmitter being matched to the same state, we derive both upper and lower bounds on the system capacity given the transmitter switching strategy through corresponding bounds on the matching probability. Using these bounds we explore the benefits and costs associated with the forward and feedback overheads. We see that in high error probability scenarios the offline scheme discussed is very close to achieving the optimal matching probability.

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