Sum Power Iterative Waterfilling for Gaussian Vector Broadcast Channels

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Abstract— **We obtain an efficient algorithm for computing the sum capacity of vector broadcast channel. This algorithm utilizes the duality between broadcast and multiple access channels and the Kuhn-Tucker conditions of sum power multiple access channel.**

I. Introduction

A *vector* broadcast channel is a one-to-many channel, where the channel between the central transmitter and the receiver i out of K receivers is given by matrix H_i. Mathematically, such a channel is given by $y_i = H_i x + n_i$, where y_i and n_i are respectively the received signal vector and the additive Gaussian noise vector at receiver i, and x is the transmitted signal vector.

Recently, an achievable region of this channel, known as the *dirty paper region*, was characterized [1], and the region was shown to achieve sum capacity by many research groups simultaneously (See journal version of [1] and the references therein). The dirty paper characterization in [1] is in terms of transmit covariance matrices $\{Q_i\}_{i=1}^K$, where Q_i corresponds to user i. This characterization, however, turns out to be non-convex in $\{Q_i\}$. In this paper, our aim is to find an efficient algorithm to compute the optimal $\{Q_i\}$ such that the sum capacity of the broadcast channel can be computed.

Simultaneously, a *duality* result was obtained in [2] showing that the capacity region of the vector multiple access channel (MAC) with sum power constraint on the transmitters is *equal* to the dirty paper achievable region. Moreover, an explicit transformation connects the optimal transmission scheme for the MAC with the $\{Q_i\}$ above. Thus, we obtain an efficient algorithm to solve the convex *dual* MAC problem given by

$$\max_{\left\{S_i : \sum_{i=1}^K \text{Tr}(S_i) \le P,\ S_i \ge 0\ \forall i\right\}} \log \left| I + \sum_{i=1}^K H_i^\dagger S_i H_i \right| \quad (1)$$

and then transform the solution using duality to obtain $\{Q_i\}$. An algorithm to compute the optimal transmit policy for a MAC with per-user power constraints on the transmitters was obtained in [3]. However, this algorithm cannot be directly applied to the dual MAC due to the difference in the power constraint.

II. The Algorithm

The iterative algorithm converges to a fixed point which satisfies the Kuhn-Tucker conditions of (1), and hence obtains the solution. Let $S_i(l)$ denote the l'th iteration of S_i. Then the algorithm can be summarized as follows:

1. Initialize covariance matrices to zero: $S_i(0) = 0\ \forall i$.
2. For iteration l : Generate *effective* channels $H_j^{eff} = H_j(I + \sum_{i \neq j}^M H_i^\dagger S_i(l-1) H_i)^{-1/2}$.
3. Treating these effective channels as parallel, non-interfering channels, obtain covariance matrices M_i by waterfilling with total power P.
   $\{M_i(l)\}_{i=1}^K = \text{argmax} \sum_{i=1}^K \log \left| I + (H_i^{eff})^\dagger A_i H_i^{eff} \right|$
   over the set $A_i \ge 0, \sum_{i=1}^K \text{Tr}(A_i) = P$
   This maximization is equivalent to waterfilling the block diagonal channel with diagonals equal to H_j^{eff}.
4. Compute the new $S_i(l) = \frac{(K-1)S_i(l-1) + M_i(l)}{K}$ for all i.
5. Return to Step 2 until desired accuracy is reached.

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III. Convergence and Optimality

Here, we provide an outline for convergence and optimality. First, convergence is considered. Define function f as below:

$$f(S_1, S_2, \cdots, S_K) = \log \left| I + \sum_{i=1}^K H_i^\dagger S_i H_i \right|$$

Then, the following can be shown.

$$\begin{aligned} K\, f(S_1(l-1), S_2(l-1), \cdots, S_K(l-1)) \\ \le f\,(M_1(l), S_2(l-1), \cdots, S_K(l-1)) \quad (2) \\ + f(S_1(l-1), M_2(l), \cdots, S_K(l-1)) \\ + \cdots + f(S_1(l-1), S_2(l-1), \cdots, M_K(l)) \\ \le K\, f\left(\frac{(K-1)S_1(l-1) + M_1(l)}{K}, \cdots \right. \quad (3) \\ \left. \cdots, \frac{(K-1)S_K(l-1) + M_K(l)}{K}\right) \end{aligned}$$

The inequality given by Equation (2) is due to the optimality of single-user waterfilling in step 3, and the inequality given by Equation (3) is due to the concavity of $f(\cdot)$ and Jensen's inequality. Jensen's inequality guarantees a strict increase of the function value when any of $S_i(l)$ is different from $S_i(l-1)$. Therefore, the function value monotonically increases. However, the function value is upper bounded, and we can conclude that the algorithm converges to a fixed point. Note that no loop can exist due to the strict inequality for $S_i(l) \neq S_i(l-1)$.

Also, S_i can be shown to converge to an optimal point. Due to the concavity of the problem, Kuhn-Tucker conditions are necessary and sufficient for optimality. The Kuhn-Tucker conditions can be derived in a similar way as in [3], and can be shown to be satisfied by the limit of $\{S_i\}$ due to steps 2 and 3 of the algorithm.

References

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