On The Duality of Multiple-Access and Broadcast Channels

Nihar Jindal, Sriram Vishwanath, and Andrea Goldsmith

Department of Electrical Engineering
Stanford University
Stanford, CA 94305
njindal, sriram, andrea@wsl.stanford.edu

Abstract

We show that the Gaussian multiple-access channel and the Gaussian broadcast channel are fundamentally related and are essentially duals of each other. The dual channels we consider have the same channel gains and the same noise power at all receivers. We show that the capacity region of a broadcast channel (both constant and fading) can be written in terms of the capacity region of the dual multiple-access channel, and vice versa. We can use this result to find the capacity region of the MAC if the capacity region of only the broadcast channel is known, and vice versa. For fading channels we show duality under ergodic capacity, but duality also holds for different notions of Shannon capacity for fading channels such as outage capacity and minimum rate capacity. Using duality, many results known for only one of the two channels are now known for the dual channel as well.

1 Introduction

Two of the most important results in network information theory are the capacity of the Gaussian multiple-access channel (MAC) and the capacity of the Gaussian broadcast channel (BC). Though both of these channels have been studied extensively, no relationship has previously been discovered between the two channels. We show that the Gaussian MAC and BC are essentially duals of each other and as a result the capacity regions of the BC and the MAC with the same channel gains (i.e. the channel gain of receiver j in the BC equals the channel gain of transmitter j in the MAC) and the same noise power at every receiver (i.e. each receiver in the BC and the single receiver in the MAC has the same noise power) are very closely related.

The Gaussian MAC and the Gaussian BC have two fundamental differences. In the MAC, *each transmitter* has an individual power constraint, whereas in the BC there is only a single power constraint on the transmitter. In addition, signal and interference come from *different* transmitters in the MAC and are therefore multiplied by *different* channel gains (known as the near-far effect) before being received, whereas in the BC all signals come from the same source and therefore have the same channel gain. Though the channels may not appear to be related, there is a striking similarity in the coding/decoding scheme used to achieve the capacity of the Gaussian MAC and BC. For the MAC, each user transmits Gaussian codewords which are scaled by the channel and then "added" in the air. Decoding is done using successive decoding, in which a certain user's codeword is decoded first and then subtracted from the received signal, and then the next user is decoded, etc. For the Gaussian BC, superposition coding, which is optimal in general for degraded broadcast channels such as the Gaussian BC, simplifies to transmitting the sum of independent Gaussian codewords (one codeword per user).

The receivers also perform successive decoding, with the catch that each user can decode and therefore subtract out only the codewords of users with smaller channel gains than themselves. In both channels, the received signal is a sum of Gaussian codewords and successive decoding is performed. The similarity in the coding/decoding process for the Gaussian MAC and BC hints at the relationship between the channels.

We first show that the capacity region of a M-user Gaussian MAC lies within the capacity region of the dual Gaussian BC with power constraint equal to the sum of the MAC power constraints and that the MAC capacity region boundary touches the BC capacity region boundary at one point. This result holds for a constant (AWGN) or fading channel and is the fundamental relationship that allows us to relate the capacity regions of the two channels. Using this basic relationship, we find an expression for the capacity region of the BC in terms of the capacity region of the dual MAC. We also find an expression for the capacity region of the MAC in terms of the capacity region of the dual BC. Furthermore, we show that there exists a direct relationship between the optimum power allocation scheme used to achieve points along the boundary of the capacity region of the dual MAC and BC. In this paper we consider flat-fading channels, but all results apply to time-invariant channels with ISI as well.

The remainder of this paper is organized as follows. In Section 2 we describe the dual Gaussian BC and MAC. In Section 3 we show that the constant Gaussian BC and MAC are duals. In Section 4, we show that fading BC and MAC are also duals. We consider some applications and extensions of this duality in Section 5, followed by our conclusions. Proofs of all theorems are omitted for brevity and are in [7].

2 System Model

The notation used in this paper is as follows: Boldface is used to denote vectors. $\mathbb{E}_H$ is used to denote expectation over the random variable H and lower case h denotes a realization of H.

We consider two different discrete time systems as shown in Fig. 1, where i denotes the time index. The system to the left is a broadcast channel: a one to many system, where the signal $X[i]$ is sent to M different receivers simultaneously. Each receiver is assumed to suffer from flat-fading, i.e. the desired signal $X[i]$ is multiplied by a channel gain[1] $\sqrt{h_j[i]}$, and white Gaussian noise $n_j[i]$ is added to the received signal. We let $\mathbf{h}[i] = (h_1[i], \ldots, h_M[i])$ denote the vector of channel gains at time i.

The system to the right is a multiple access channel: a many to one system, where M different transmitters send different signals $X_j[i]$ to a single receiver. The signals sent by each user experience flat fading and their sum is obtained at the receiver. The receiver is assumed to have additive white Gaussian noise $n[i]$.

Mathematically, the two systems can be described as:

$$\text{BC: } Y_j[i] = \sqrt{h_j[i]}X[i] + n_j[i] \qquad\qquad \text{MAC: } Y[i] = \sum_{j=1}^{M} \sqrt{h_j[i]}X_j[i] + n[i].$$

Notice that in our system models, the noise power of all the receivers in the BC and the single receiver in the MAC are equal to σ^2. Also, the term $\sqrt{h_j[i]}$ is the channel gain of receiver j in the BC (downlink) and $\sqrt{h_j[i]}$ is *also* the channel gain of transmitter j in the MAC (uplink). We call this BC the *dual* of the MAC, and vice versa.

[1] In general the channel gain may be complex, but assuming perfect phase information at the receivers, without loss of generality we consider real channel gains only.

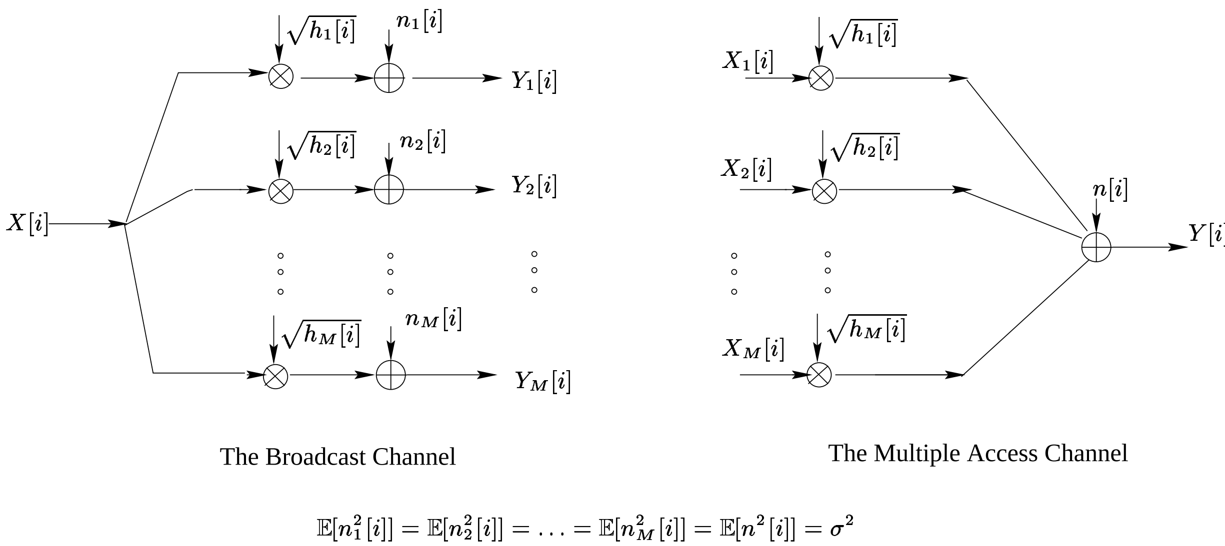


Figure 1: System Models

We consider two different models in this paper: constant and fading channel gains. In the constant channel, the channel gains $h_j[i]$ are constant for all i and these values are assumed to be known at all the transmitters and the receivers in the MAC and BC. In the fading channel, the channel gains $(H_1[i], \ldots, H_M[i])$ are a jointly stationary and ergodic random process. There are many different assumptions that can be made about the channel state information (CSI) known at the transmitters and receivers. In this paper we assume perfect CSI, or that all transmitters and receivers know $\mathbf{h}[i]$ perfectly at time i.

3 Duality of the Constant MAC and BC

We now state the fundamental relationship between the constant MAC and BC. In Sections 3 and 4 we present results on the two-user channel, but these also extend to the general M-user case.

Theorem 1 *The capacity region of a constant Gaussian MAC is a subset of the capacity region of the dual BC with power constraint equal to the sum of the MAC power constraints:*

$$\mathcal{C}_{MAC}(P_1, P_2; h_1, h_2) \subseteq \mathcal{C}_{BC}(P_1 + P_2; h_1, h_2) \tag{1}$$

Furthermore, the boundaries of the two regions intersect at one point.

Fig. 2 shows the relationship between the capacity regions of the dual MAC and BC described in Theorem 1. Notice the MAC region is a subset of the BC region and their boundaries meet at exactly one point. If we were to vary P_1 and P_2 such that their sum did not change, we would get a *different* MAC capacity region that intersects the BC boundary at a *different* point.

All boundary points of the BC capacity region are achieved using successive decoding in which the stronger user (i.e. the user with the larger channel gain) is decoded last. In the MAC, a different decoding order is used for different boundary points. We have found, however, that the MAC and BC *always* intersect at the corner point of the MAC region corresponding to decoding the weaker user last. This is *opposite* the decoding order of the BC.

If we equate the rates of the two channels at the intersection of the MAC and BC boundaries, we can find a relationship between the powers needed in the dual BC (P_1^B, P_2^B) and MAC

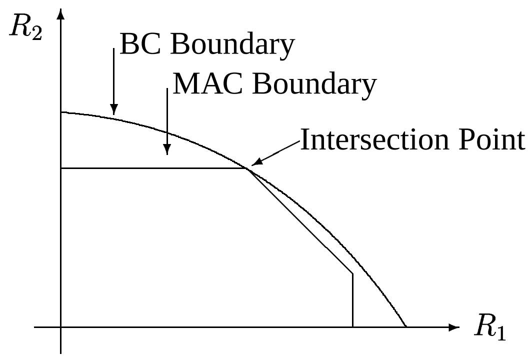


Figure 2: Capacity regions for a constant MAC and its dual BC with $h_1 > h_2$.

(P_1^M, P_2^M) to achieve the same rates:

$$R_1^M = \frac{1}{2}\log\left(1 + \frac{h_1 P_1^M}{h_2 P_2^M + \sigma^2}\right) = \frac{1}{2}\log\left(1 + \frac{h_1 P_1^B}{\sigma^2}\right) = R_1^B \tag{2}$$

$$R_2^M = \frac{1}{2}\log\left(1 + \frac{h_2 P_2^M}{\sigma^2}\right) = \frac{1}{2}\log\left(1 + \frac{h_2 P_2^B}{h_2 P_1^B + \sigma^2}\right) = R_2^B. \tag{3}$$

The powers (P_1^M, P_2^M) and (P_1^B, P_2^B) are then related by the *duality power transformation*:

$$\begin{aligned} P_1^B &= \frac{P_1^M \sigma^2}{h_2 P_2^M + \sigma^2}, \quad P_2^B = \frac{P_2^M (h_2 P_1^B + \sigma^2)}{\sigma^2} \\ P_1^M &= \frac{P_1^B (h_2 P_2^M + \sigma^2)}{\sigma^2}, \quad P_2^M = \frac{P_2^B \sigma^2}{h_2 P_1^B + \sigma^2} \\ P_1^M + P_2^M &= P_1^B + P_2^B. \end{aligned} \tag{4}$$

This transformation tells us that any point along the boundary of the BC capacity region is achievable in the MAC using the same *sum* power. In other words, given some powers (P_1^B, P_2^B) in the BC (and therefore optimal rates R_1^B and R_2^B), we can find (P_1^M, P_2^M) such that the same rates are achievable in the MAC with $P_1^M + P_2^M = P_1^B + P_2^B$. The inverse relationship also is true: any rates achievable in the MAC are also achievable in the BC with the same sum power.

3.1 Multiple-Access Channel To Broadcast Channel

In this section we show that the capacity region of a Gaussian BC can be characterized in terms of the capacity region of the dual MAC.

Theorem 2 *The capacity region of a constant Gaussian BC with power P is equal to the union of capacity regions of the dual MAC with power P_1 and P_2 such that $P_1 + P_2 = P$:*

$$\mathcal{C}_{BC}(P; h_1, h_2) = \bigcup_{0 \le P_1 \le P} \mathcal{C}_{MAC}(P_1, P - P_1; h_1, h_2). \tag{5}$$

An example of this theorem is illustrated in Fig. 3(a), where $\mathcal{C}_{MAC}(P_1, P - P_1; h_1, h_2)$ is plotted for different values of P_1. The BC capacity region is shown by the bold line. Notice that each MAC boundary touches the BC boundary at a *different* point. More generally, each point along the BC boundary is intersected by a different MAC boundary and the BC and MAC power schemes are related by the duality power transformation in (4)

If we look more carefully at the union expression in (5), we notice that the union of MAC's is nothing more than the capacity region of the MAC with a *sum* power constraint P instead of *individual* power constraints P_1 and $P - P_1$. This is the channel where the two transmitters are not allowed to transmit cooperatively but are allowed to draw from a common source of power (i.e. the sum of the transmit power is bounded instead of individual bounds on transmit power). Theorem 2 therefore implies that *the capacity region of the MAC with sum power constraint P equals the capacity region of the dual BC with power constraint P.* If the MAC is allowed to have a sum power constraint, similar to the BC power constraint, the channels are equivalent. This is a very surprising result considering the significant differences between the MAC and BC.

3.2 Broadcast Channel to Multiple Access Channel

In this section, we show that the capacity region of the MAC can be characterized in terms of the capacity region of the dual BC. In order to find this relationship, we make use of a concept called *channel scaling*. Since h_j and P_j always appear as a product in the constant MAC capacity expressions, we can arbitrarily scale h_j by a constant and scale P_j by the inverse of the constant and still get the same capacity region. Specifically,

$$\begin{aligned} \mathcal{C}_{MAC}(P_1, P_2; h_1, h_2) &= \left\{ \mathbf{R} : \sum_{j \in S} R_j \leq \frac{1}{2} \log \left(1 + \frac{1}{\sigma^2} \sum_{j \in S} h_j P_j \right), \quad \forall S \subset \{1, 2\} \right\} \\ &= \mathcal{C}_{MAC}\left(\frac{P_1}{\alpha_1}, \frac{P_2}{\alpha_2}; \alpha_1 h_1, \alpha_2 h_2\right) \quad \forall \alpha_1, \alpha_2 > 0. \end{aligned} \tag{6}$$

In contrast, the capacity region of the scaled BC channel is different for different values of α_1 and α_2 since the BC capacity and optimal decoding order depends on the channel gains. By Theorem 1 we know $\mathcal{C}_{MAC}(\frac{P_1}{\alpha_1}, \frac{P_2}{\alpha_2}; \alpha_1 h_1, \alpha_2 h_2) \subseteq \mathcal{C}_{BC}(\frac{P_1}{\alpha_1} + \frac{P_2}{\alpha_2}; \alpha_1 h_1, \alpha_2 h_2)$. By combining this relationship with the observation in (6), we find that the unscaled MAC capacity region is a subset of the scaled BC capacity region *for all scalings:*

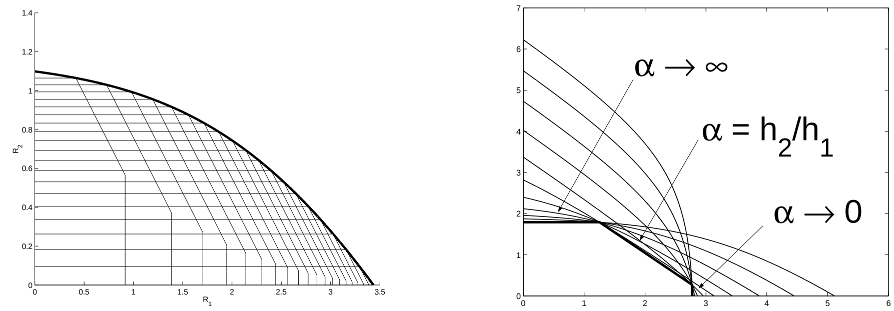
$$\mathcal{C}_{MAC}(P_1, P_2; h_1, h_2) \subseteq \mathcal{C}_{BC}\left(\frac{P_1}{\alpha_1} + \frac{P_2}{\alpha_2}; \alpha_1 h_1, \alpha_2 h_2\right) \quad \forall \alpha_1, \alpha_2 > 0 \tag{7}$$

and the boundaries of the MAC capacity region and each scaled BC capacity region intersect. If we let $\alpha = \frac{\alpha_1}{\alpha_2}$, we find that we only need to scale the channel of User 1. The scaling relationship in (7) allows us to define the MAC capacity region in terms of the dual BC.

Theorem 3 *The capacity region of a constant Gaussian MAC is equal to the intersection of the capacity regions of the scaled dual BC over all scalings:*

$$\mathcal{C}_{MAC}(P_1, P_2; h_1, h_2) = \bigcap_{\alpha > 0} \mathcal{C}_{BC}\left(\frac{P_1}{\alpha} + P_2; \alpha h_1, h_2\right). \tag{8}$$

Theorem 3 is illustrated in Figure 3(b), where we show $\mathcal{C}_{BC}(P_1/\alpha + P_2; \alpha h_1, h_2)$ for a range of values of α. Note that since the constant MAC region is a pentagon, the BC channels characterized by $\alpha = 0$, $\alpha = h_2/h_1$ and $\alpha = \infty$ are sufficient to form the pentagon. If $\alpha = h_2/h_1$, the channel gains of both users are the same and the BC capacity region is bounded by a straight line segment because the capacity region can be achieved by time-sharing. This line segment corresponds exactly with the forty-five degree line bounding the MAC capacity



(a) Constant MAC to BC

(b) Constant BC to MAC

Figure 3: Duality of the constant MAC and BC

region. As $\alpha \to 0$, the total amount of power becomes infinite but the channel gains of user 1 go to zero as well. These effects negate each other and cause $R_1 \to \log(1 + \frac{h_1 P_1}{\sigma^2})$ and $R_2 \to \infty$. As $\alpha \to \infty$, the total amount of power goes to P_2 and the channel gain of user 1 becomes infinite. These cause $R_1 \to \infty$ and $R_2 \to \log(1 + \frac{h_2 P_2}{\sigma^2})$. These two regions bound the vertical and horizontal line segments, respectively, of the MAC capacity region. All scaled BC capacity regions except the $\alpha = h_2/h_1$ channel intersect the MAC at exactly one of the two corner points of the MAC region. The $\alpha = h_2/h_1$ channel intersects the MAC region all along the forty-five degree line bounding the MAC capacity region.

4 Duality of the Fading MAC and BC

In this section we show that duality holds for the ergodic capacity regions (subject to an average power constraint) of the dual fading MAC and BC. We first formally define the ergodic capacity regions of the fading MAC and BC.

4.1 Ergodic Capacity Region of the Broadcast Channel

We define a power policy $\mathcal{P}$ over all possible fading states as a function that maps from a joint fading state $\mathbf{h} = (h_1, \ldots, h_M)$ to the transmitted power $P_j(\mathbf{h})$ for each user for all $\mathbf{h}$. Let $\mathcal{F}_{BC}$ denote the set of all power policies satisfying the average power constraint, or $\mathcal{F}_{BC} = \{\mathcal{P} : \mathbb{E}_{\mathbf{H}}[\sum_{j=1}^{M} P_j(\mathbf{h})] \le \overline{P}\}$.

From Theorem 1 of [1], the ergodic capacity region of the BC with perfect CSI and power constraint $\overline{P}$ is the union over all power policies in $\mathcal{F}_{BC}$:

$$\mathcal{C}_{BC}(\overline{P}; H_1, \ldots, H_M) = \bigcup_{\mathcal{P} \in \mathcal{F}_{BC}} \mathcal{C}_{BC}(\mathcal{P}; H_1, \ldots, H_M)$$

where

$$\mathcal{C}_{BC}(\mathcal{P}; H_1, \ldots, H_M) = \left\{ \mathbf{R} : R_j \le \mathbb{E}_{\mathbf{H}} \left[\frac{1}{2} \log \left(1 + \frac{h_j P_j(\mathbf{h})}{\sigma^2 + h_j \sum_{k=1}^{M} P_k(\mathbf{h}) \mathbf{1}[h_k > h_j]} \right) \right], \quad j = 1, \ldots, M \right\}.$$

The optimal power policies are the policies that achieve points on the boundary of $\mathcal{C}_{BC}(\overline{P}; H_1, \ldots, H_M)$ and every point on the boundary is achieved using a *different* optimal power policy.

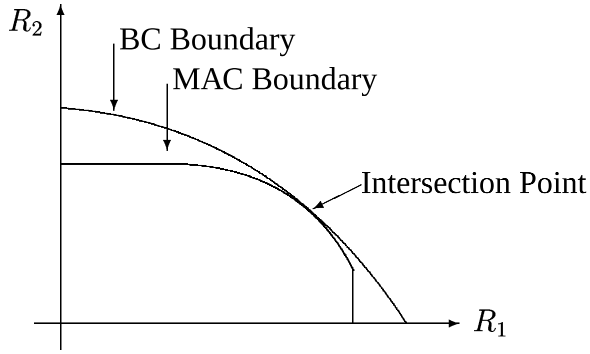


Figure 4: Capacity regions for the dual fading MAC and BC

4.2 Ergodic Capacity Region of the Multiple-Access Channel

This region is defined using the same notion of a power policy as used for the BC. Specifically, we let $\mathcal{F}_{\mathcal{MAC}}$ denote the set of all power policies satisfying the M individual power constraints, or $\mathcal{F}_{MAC} = \{\mathcal{P} : \mathbb{E}_{\mathbf{H}}[P_j(\mathbf{h})] \leq \overline{P}_j \quad 1 \leq j \leq M\}$.

From Theorem 2.1 of [4], the ergodic capacity region of the multiple-access channel with perfect CSI and power constraints $(\overline{P}_1, \dots, \overline{P}_M)$ is:

$$\mathcal{C}_{MAC}(\overline{P}_1, \dots, \overline{P}_M; H_1, \dots, H_M) = \bigcup_{\mathcal{P} \in \mathcal{F}_{MAC}} \mathcal{C}_{MAC}(\mathcal{P}; H_1, \dots, H_M)$$

where

$$\mathcal{C}_{MAC}(\mathcal{P}; H_1, \dots, H_M) = \left\{ \mathbf{R} : \sum_{j \in S} R_j \leq \mathbb{E}_{\mathbf{H}} \left[\frac{1}{2} \log \left(1 + \frac{1}{\sigma^2} \sum_{j \in S} h_j P_j(\mathbf{h}) \right) \right], \forall S \subset \{1, \dots, M\} \right\}$$

and S is any subset of $\{1, \dots, M\}$. The optimal MAC power policies are the policies which achieve points on the boundary of $\mathcal{C}_{MAC}(\overline{P}_1, \dots, \overline{P}_M; H_1, \dots, H_M)$ and every point on the boundary is achieved using a *different* optimal power policy.

4.3 Subset Relationship of the Fading MAC and BC

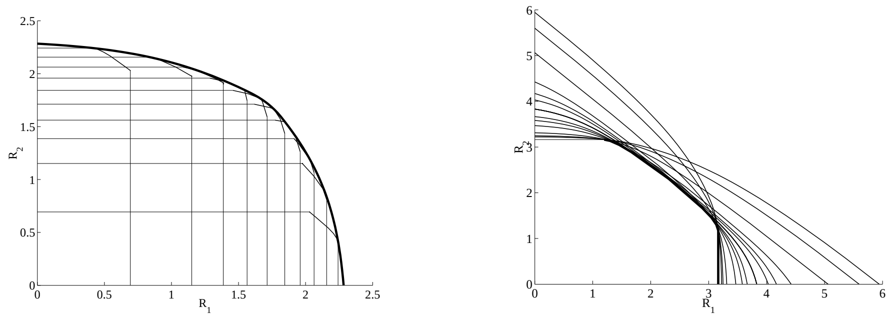
We now relate the ergodic capacity regions of the dual MAC and BC.

Theorem 4 *The ergodic capacity region of a fading 2-user Gaussian MAC is a subset of the ergodic capacity region of the dual BC with power constraint equal to the sum of the MAC power constraints:*

$$\mathcal{C}_{MAC}(\overline{P}_1, \overline{P}_2; H_1, H_2) \subseteq \mathcal{C}_{BC}(\overline{P}_1 + \overline{P}_2; H_1, H_2) \tag{9}$$

Furthermore, the boundaries of the two regions intersect at least once.

Fig. 4 illustrates the subset relationship in (9) of the capacity regions of the dual fading MAC and BC. Due to the fading, the ergodic capacity region of the MAC is bounded by straight line segments connected by a curved section as opposed to the pentagon we saw for the constant MAC. The BC and MAC intersect in the curved section of the MAC boundary. Again, if we were to vary $\overline{P}_1$ and $\overline{P}_2$ such that their sum didn't change, we would get a *different* MAC ergodic capacity region that intersects the BC ergodic capacity region boundary at a *different* point.



(a) Fading MAC to BC

(b) Fading BC to MAC

Figure 5: Duality of the fading MAC and BC

4.4 Multiple-Access Channel To Broadcast Channel

We now characterize the ergodic capacity region of the BC in terms of the dual MAC.

Theorem 5 *The ergodic capacity region of a fading Gaussian BC with power constraint $\overline{P}$ is equal to the union of ergodic capacity regions of the dual MAC with power constraints $\overline{P}_1$ and $\overline{P}_2$ such that $\overline{P}_1 + \overline{P}_2 = \overline{P}$:*

$$\mathcal{C}_{BC}(\overline{P}; H_1, H_2) = \bigcup_{0 \leq \overline{P}_1 \leq \overline{P}} \mathcal{C}_{MAC}(\overline{P}_1, \overline{P} - \overline{P}_1; H_1, H_2). \tag{10}$$

Fig. 5(a) illustrates this theorem. As in Fig. 3(a), each MAC boundary intersects the BC boundary at a different point. The MAC and BC optimal power policies that achieve the point where the MAC and BC boundaries intersect are related by the duality power transformation in (4). By applying the duality power transformation *in each fading state* to a BC power policy, we can find a MAC power policy that achieves the same user-by-user rates in each fading state and uses the same *sum* power in each fading state. If the rates and sum power are the same in each fading state, then the average rates and average sum power will also be equivalent and therefore the same average rates are achievable in the MAC. Since the duality power transformation works in both way (i.e. MAC to BC and BC to MAC), the inverse statement is also true: for any MAC power policy, a BC power policy can be found that achieves the same rates with the same sum power. As we saw for constant channels, we find that the ergodic capacity region of the MAC with a *sum* power constraint P equals the ergodic capacity region of the dual BC with power constraint P.

In addition to finding the the ergodic capacity region of the MAC or BC in terms of its dual channel's ergodic capacity region, *duality also allows us to find the optimal power policies of either the MAC or BC in terms of the optimal power policies of its dual*. Notice that the decoding order in each fading state must be reversed when using the duality power transformation. From results on general degraded broadcast channels, we know that the stronger user is always decoded last in the BC, but by the duality power transformation we discover that along the boundary of the MAC capacity region, *the weaker user is always decoded last in the MAC*. This may not seem to be consistent with results on the MAC ergodic capacity region that state that the decoding should be done in order of increasing priorities [4], but this is resolved by the fact that in optimal power policies only one of the two users transmit (which means decoding order is irrelevant) in fading states in which this inconsistency arises. This concept is discussed in more detail in [7].

4.5 Broadcast Channel to Multiple Access Channel

In order to characterize the ergodic capacity region of the MAC in terms of the dual BC, we must use channel scaling again. Fortunately, the channel scaling relationship in (7) also holds for fading channels.

Theorem 6 *The ergodic capacity region of a fading MAC channel is equal to the intersection of the ergodic capacity regions of the scaled dual BC over all scalings:*

$$\mathcal{C}_{MAC}(\overline{P}_1, \overline{P}_2; H_1, H_2) = \bigcap_{\alpha>0} \mathcal{C}_{BC}(\frac{\overline{P}_1}{\alpha} + \overline{P}_2; \alpha H_1, H_2). \tag{11}$$

An illustration of this is given in Figure 5(b). Every different scaled BC ergodic capacity region intersects the MAC ergodic capacity region at a *different* point. Again, the optimal power policies at these intersection points are related by the duality power transformation. The MAC ergodic capacity region cannot be characterized by only three BC regions as it was for the constant channel case. The BC regions where $\alpha \to 0$ and $\alpha \to \infty$ still limit the vertical and horizontal line segments of the MAC ergodic capacity region. The curved section of the MAC boundary, however, is intersected by many different scaled BC ergodic capacity regions.

5 Applications and Extensions

So far we have shown that the duality of the MAC and BC holds for capacity of constant channels and ergodic capacity of fading channels. However, the duality of the Gaussian MAC and BC is actually a more general result and also holds for alternative definitions of Shannon capacity and for multiple-antenna Gaussian channels. Outage capacity [2,3,5] and minimum rate capacity [6] are two notions of Shannon capacity for which duality also holds. These definitions of capacity have restrictions on the *instantaneous* transmission rates (i.e. the state-by-state rates) which are not present in ergodic capacity, but because the duality power transformation preserves state-by-state rates (i.e. the same instantaneous rates are achieved in the dual channels), all results seen to hold for ergodic capacity (Theorems 4 - 6) also hold for minimum rate and outage capacity. The minimum rate capacity region for the fading BC is derived in [6]. Using duality and this result, the minimum rate capacity region, along with the optimal power allocation scheme and decoding order, for the fading MAC is found in [7].

We have also found that the multiple-input, multiple-output (MIMO) Gaussian BC and MAC are duals [9]. Since the MIMO BC is a non-degraded broadcast channel, it is very difficult to find its capacity region. The capacity region of the MIMO MAC, however, has been found [8]. Using the duality of the channels, we have shown that any rate achievable in the MIMO MAC is also achievable in the MIMO BC using the same sum power. It still remains to be seen if this achievable region is actually the capacity region of the MIMO BC or if it is only a lower bound of the true capacity region.

Though we have addressed only the Gaussian MAC and BC, it may be possible that some form of duality holds for arbitrary multi-terminal Gaussian networks, such as the two sender, two receiver channel. Due to the lack of a general theory about multi-terminal Gaussian networks, verifying such a duality may be very far off or even impossible. It is also possible that a duality exists between non-Gaussian broadcast and multiple-access channels. We have found that a duality exists between a very limited set of non-Gaussian broadcast and multiple-access channels, but it is still unclear if duality holds in general for such channels.

6 Conclusion

We have shown that the Gaussian MAC and BC are dual channels. Using the fact that the MAC capacity region is a subset of the dual BC capacity region and that the boundaries intersect at one point, we were able to write expressions for the capacity of both the MAC and the BC in terms of the dual channel capacity region. We also saw that the capacity of the MAC with a sum power constraint instead of individual transmitter power constraints is equal to the capacity of the dual BC with the same power constraint. Using duality, results that are known for one of the two channels can often be transferred to the other channel, i.e. if the Shannon capacity with certain constraints is known for the BC, then the capacity with the same constraints can easily be found for the MAC as well. In addition, the optimal power allocation schemes for the two channels are related through a simple state-by-state power transformation which preserves the rates achieved in each fading state and total sum power. We also saw that the MAC and BC require opposite decoding orders to achieve boundary points of the capacity region. In summary, the duality of the Gaussian BC and MAC is not only a very interesting theoretical result, but it is also of great practical significance because problems that are solved for only one of the two channels are solved for the dual channel as well.

References

[1] L. Li and A. Goldsmith, “Capacity and optimal resource allocation for fading broadcast channels: Part I: Ergodic capacity”, IEEE Trans. Inform. Theory, vol. 47, pp 1083-1102, March 2001.

[2] L. Li and A. Goldsmith, “Capacity and optimal resource allocation for fading broadcast channels: Part II: Outage capacity”, IEEE Trans. Inform. Theory, vol. 47, pp 1103-1127, March 2001.

[3] L. Li and A. Goldsmith, “Outage capacities and optimal power allocation for fading multiple-access channels”, in revision for IEEE Trans. Inform. Theory.

[4] D.N. Tse and S. Hanly, ”Multi-Access Fading Channels: Part I: Polymatroid Structure, Optimal Resource Allocation and Throughput Capacities”, IEEE Trans. Inform. Theory, v. 44, pp 2796-2815, Nov. 1998.

[5] S.V. Hanly and D. N. Tse, “Multiaccess fading channels-Part II:Delay-limited capacities”, IEEE Trans. Inform. Theory, vol. 44, pp. 2816-2831, Nov. 1998.

[6] N. Jindal and A. Goldsmith, “Capacity and optimal power allocation for fading broadcast channels with minimum rates”, To appear: Proceedings: IEEE Globecom, Nov. 2001.

[7] N. Jindal, S. Vishwanath, and A. Goldsmith, “On the Duality of Multiple-Access and Broadcast Channels”, in preparation for journal submission.

[8] W. Yu, W. Rhee, S. Boyd, J. Cioffi, “Iterative water-filling for vector multiple access channels”, IEEE International Symposium on Information Theory, (ISIT), 2001.

[9] S. Vishwanath, N. Jindal, and A. Goldsmith, “On the Capacity of Multiple Input Multiple Output Broadcast Channels”, Submitted to ICC, 2002.