A Semidefinite Relaxation Approach to MIMO Detection for High-Order QAM Constellations

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***Abstract*—A new and conceptually simple semidefinite relaxation approach is proposed for MIMO detection in communication systems employing high-order QAM constellations. The new approach affords improved detection performance compared to existing solutions of comparable worst-case complexity order, which is nearly cubic in the dimension of the transmitted symbol vector and independent of the constellation order for uniform QAM, or affine in the constellation order for nonuniform QAM.**

***Index Terms*—High-order QAM, integer least squares, lattice search, multiple-input multiple-output (MIMO) detection, multiuser detection, semidefinite programming, semidefinite relaxation (SDR).**

I. Introduction

Maximum-likelihood (ML) detection in memoryless multiple-input multiple-output (MIMO) communication systems with Gaussian noise is equivalent to a least-squares lattice search problem that is NP-hard. For this reason, several computationally efficient approximate solutions have been developed. The current state-of-the-art includes two main families of high-performance MIMO detectors: those based on *sphere decoding* (SD) [1], [2], [11], [12], [14] and those based on *semidefinite relaxation* (SDR) [5]–[7], [13]. SD detectors can provide the exact ML solution at low computational cost, provided that the signal-to-noise ratio (SNR) is relatively high, and the aggregate transmission rate is relatively low. However, SD cannot efficiently handle high problem dimensions (long symbol vectors) or high-order symbol constellations, especially at low SNR, and it has recently been shown that its *expected complexity* is exponential [4], under certain conditions that are relatively mild and general in our context. Worst-case complexity of computing the exact ML solution is generically exponential, due to NP-hardness.

In contrast, SDR approaches feature polynomial *worst-case* complexity and very competitive performance. Initially, SDR multiuser/MIMO detection was developed for binary phase-shift keying (BPSK) constellations, but the ideas were later extended to M-PSK [5]–[7] and, very recently, to 16-quadrature amplitude modulation (16-QAM) [13] and general QAM constellations [8]. While [13] deals exclusively with 16-QAM, the approach can, in principle, be extended to higher-order QAM alphabets. This, however, entails the introduction of additional slack variables, and complexity becomes $O(K^{6.5}N^{6.5})$, where $N = O(M)$, M is the number of symbols, and K is the square root of the order of the constellation. The idea in [13] is fruitful for 16-QAM but impractical for higher orders. Likewise, the complexity of the methods in [8] ranges from $O(K^{6.5}N^{4})$ to $O(K^{6.5}N^{6.5})$.

In this letter, we propose a different, $O(N^{3.5})$ relaxation for high-order QAM alphabets. Our approach can be viewed as further relaxation of [13], only utilizing upper and lower bounds on the symbol energy in the relaxation step. The key features of our approach are that 1) it provides significant performance improvements relative to existing solutions of comparable worst-case complexity order; and 2) its complexity is independent of the constellation order for uniform QAM and affine in the constellation order for nonuniform QAM. For BPSK and 4-QAM, our approach reduces to the one in [7].

II. Problem Statement and Preliminaries

For any separable QAM constellation,[1] ML detection in memoryless MIMO communication systems with Gaussian noise can be formulated as the following optimization problem (possibly after noise prewhitening):

$$\min \|\mathbf{d} - \mathbf{M}\mathbf{s}\|_2^2 \tag{1}$$

$$\text{subject to} : \mathrm{Re}\{\mathbf{s}(i)\} \in \mathcal{A}_{\mathrm{real}}, \mathrm{Im}\{\mathbf{s}(i)\} \in \mathcal{A}_{\mathrm{imag}}, \forall i. \tag{2}$$

For brevity of exposition, we will assume that $\mathcal{A}_{\mathrm{real}} = \mathcal{A}_{\mathrm{imag}} = \mathcal{A}$ in the sequel, although our approach generalizes trivially to different alphabets for the real and imaginary parts. We thus consider

$$\min \|\mathbf{d} - \mathbf{M}\mathbf{s}\|_2^2 \tag{3}$$

$$\text{subject to} : \mathrm{Re}\{\mathbf{s}(i)\} \in \mathcal{A}, \mathrm{Im}\{\mathbf{s}(i)\} \in \mathcal{A}, \forall i \tag{4}$$

where $\mathbf{d}$ is the complex baseband received vector, $\mathbf{M}$ is a known baseband-equivalent channel matrix, and $\mathbf{s}$ is the symbol vector. Upon defining

$$\mathbf{z} := \left[\mathrm{Re}\{\mathbf{d}\}^T \mathrm{Im}\{\mathbf{d}\}^T\right]^T \tag{5}$$

[1]Separable constellations are almost always adopted for ease of decoding, even in the single-input single-output case.

Manuscript received November 10, 2005; revised February 27, 2006. The work of N. D. Sidiropoulos was supported in part by the U.S. ARO under ERO Contract N62558-03-C-0012, in part by the European Union under FP6 U-BROAD STREP # 506790, and in part by the GSRT. The work of Z.-Q. Luo was supported in part by the National Science Foundation under Grant DMS-0312416. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Philip Schniter.

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Digital Object Identifier 10.1109/LSP.2006.874443

$$\mathbf{H} := \begin{bmatrix} \mathrm{Re}\{\mathbf{M}\} & -\mathrm{Im}\{\mathbf{M}\} \\ \mathrm{Im}\{\mathbf{M}\} & \mathrm{Re}\{\mathbf{M}\} \end{bmatrix} \tag{6}$$

$$\mathbf{r} := \left[\mathrm{Re}\{\mathbf{s}\}^T \mathrm{Im}\{\mathbf{s}\}^T\right]^T \tag{7}$$

we may convert the problem to real-valued form

$$\min \|\mathbf{z} - \mathbf{H}\mathbf{r}\|_2^2 \tag{8}$$

$$\text{subject to : } \mathbf{r}(i) \in \mathcal{A},\ \forall i. \tag{9}$$

III. Proposed Solution

Assume that $\mathcal{A}$ is symmetric about the origin (always the case for QAM constellations). In this case, if $\mathbf{r}$ satisfies the finite alphabet constraints in (9), then so does $t\mathbf{r}$, for $t \in \{-1, 1\}$. Furthermore

$$\|\mathbf{z} - \mathbf{H}\mathbf{r}\|_2^2 = \mathbf{r}^T\mathbf{H}^T\mathbf{H}\mathbf{r} - 2\mathbf{z}^T\mathbf{H}\mathbf{r} + \mathbf{z}^T\mathbf{z}. \tag{10}$$

It follows that the minimization in (8) subject to (9) is equivalent to

$$\min(\mathbf{r}^T\mathbf{H}^T\mathbf{H}\mathbf{r} - 2\mathbf{z}^T\mathbf{H}t\mathbf{r}) \tag{11}$$

$$\text{subject to : } \mathbf{r}(i) \in \mathcal{A},\ \forall i,\ t \in \{-1, 1\}. \tag{12}$$

Further defining $\mathbf{x} := [\mathbf{r}^T\ t]^T \in \mathbb{R}^N$ and

$$\mathbf{Q} := \begin{bmatrix} \mathbf{H}^T\mathbf{H} & -\mathbf{H}^T\mathbf{z} \\ -\mathbf{z}^T\mathbf{H} & 0 \end{bmatrix} \tag{13}$$

the minimization of (11) subject to (12) can be put in homogeneous quadratic form

$$\min \mathbf{x}^T\mathbf{Q}\mathbf{x} \tag{14}$$

$$\text{subject to : } \mathbf{x}(i) \in \mathcal{A}, \forall i \in \{1, \cdots, N-1\}$$
$$\mathbf{x}(N) \in \{-1, 1\}. \tag{15}$$

Using $\mathbf{x}^T\mathbf{Q}\mathbf{x} = \mathrm{Trace}(\mathbf{x}^T\mathbf{Q}\mathbf{x}) = \mathrm{Trace}(\mathbf{Q}\mathbf{x}\mathbf{x}^T)$, and denoting $\mathbf{X} := \mathbf{x}\mathbf{x}^T$, we can *equivalently* rewrite the minimization in (14) under (15) as

$$\min \mathrm{Trace}(\mathbf{Q}\mathbf{X}) \tag{16}$$

$$\text{subject to : } \mathbf{X} \geq \mathbf{0},\ \mathrm{rank}(\mathbf{X}) = 1, \tag{17}$$

$$\mathbf{X}(i,i) \in \mathcal{A}^2, \forall i \in \{1, \cdots, N-1\}, \mathbf{X}(N,N) = 1 \tag{18}$$

where $\mathcal{A}^2$ contains the squared alphabet values. Problem (16)–(18) entails nonconvex constraints: the $\mathrm{rank}(\mathbf{X}) = 1$ constraint, as well as the finite (squared) alphabet constraints $\mathbf{X}(i,i) \in \mathcal{A}^2$, $\forall i \in \{1, \cdots, N-1\}$. Dropping the rank-one constraint, and relaxing the constraints $\mathbf{X}(i,i) \in \mathcal{A}^2$, $\forall i \in \{1, \cdots, N-1\}$ to the convex half-space constraints $L := \min_{a \in \mathcal{A}} a^2 \leq \mathbf{X}(i,i) \leq \max_{a \in \mathcal{A}} a^2 =: U$, $\forall i \in \{1, \cdots, N-1\}$, we obtain the following convex relaxation:

$$\min \mathrm{Trace}(\mathbf{Q}\mathbf{X}) \tag{19}$$

$$\text{subject to : } \mathbf{X} \geq \mathbf{0} \tag{20}$$

$$L \leq \mathbf{X}(i,i) \leq U, \forall i \in \{1, \cdots, N-1\}$$
$$\mathbf{X}(N,N) = 1. \tag{21}$$

Note that (19)–(21) is not a Lagrangian relaxation of (16)–(18), because, in addition to the rank-one constraint, we have relaxed the alphabet constraints. This means that the bi-dual interpretation does not hold for our relaxation in (19)–(21). For a bi-dual relaxation, see [13]. Our proposed relaxation in (19)–(21) can be viewed as further relaxation of [13], and it affords lower complexity for large $|\mathcal{A}|$ compared to [13].

The relaxed problem in (19)–(21) can be solved using any of the available modern SDP solvers, such as SeDuMi [10], based on interior point methods. After this step, an approximate solution to the original problem can be generated using *Gaussian randomization*: that is, drawing random vectors $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{X}_o)$, where $\mathbf{X}_o$ denotes the solution of (19)–(21), quantizing each element of $\mathbf{x}$ to the nearest point in $\mathcal{A}$, reconstructing $\mathbf{s}$ from the quantized $\mathbf{x}$, and picking the $\mathbf{s}$ that yields the smallest cost in (3).

A. Complexity

The worst-case complexity of solving a generic SDP problem involving a matrix variable of size $N \times N$ and $O(N)$ linear constraints is $O(N^{6.5})$. That would imply a complexity of $O(N^{6.5})$ for problem (19)–(21). However, exploiting the fact that the constraints in (21) are separable and only apply to the diagonal elements of $\mathbf{X}$, that figure can be reduced to $O(N^{3.5})$, which is very competitive ($N = 2M + 1$, where M is the number of QAM symbols). The complexity of the randomization step is $O(N^2)$ per draw. We emphasize that, unlike [13], the complexity of the overall algorithm is independent of the constellation order for uniform QAM and affine in the constellation order for nonuniform QAM. This is because the quantization step in the randomization loop amounts to simple scaling and rounding for uniform constellations but may require a linear search for nonuniform constellations.

IV. Simulations

We conducted Monte Carlo (MC) simulation experiments for two indicative MIMO transmission scenarios: a 16×16 system using 64-QAM and an 8×8 system using 16-QAM. In both cases, the channel matrix comprised i.i.d. elements drawn from a circularly symmetric zero-mean complex normal distribution of unit variance ($\mathcal{CN}(0,1)$), and a new channel realization was drawn for each vector transmission (MC trial). The SNR is defined as $\mathrm{SNR} := 10\log_{10}(ME_s/N_o)$, where M is the length of the transmitted QAM symbol vector $\mathbf{s}$, E_s is the mean symbol energy of the QAM constellation, and the noise vector is i.i.d. $\mathcal{CN}(0, N_o)$.

In order to gauge performance as a function of the number of randomizations, we tested our SDR algorithm with 100, 300, and 1000 randomization samples per decoded vector. As baselines for comparison, we employed 1) the Schnor–Euchner variant of SD (SE-SD) with an infinite radius so that the optimal solution is always obtained and 2) two commonly used suboptimal solutions of complexity $O(M^3)$: the quantized output of the zero-forcing linear receiver (QZF) and the (nonlinear) block MMSE-DFE (BMMSE-DFE) [3], [9]. Two performance

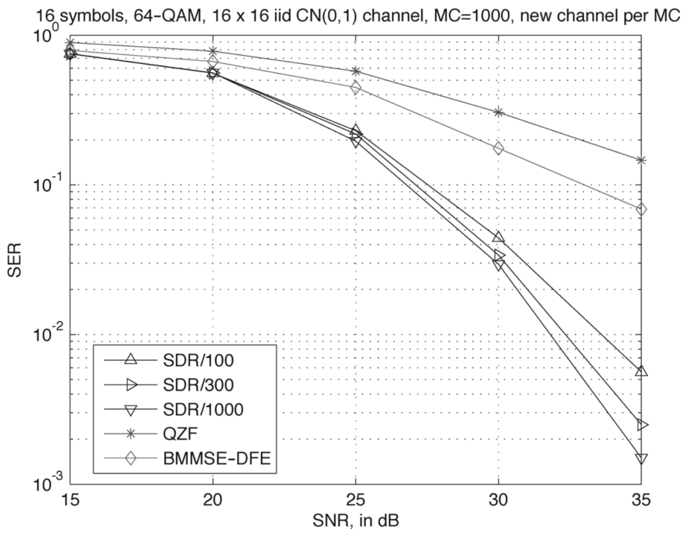

Fig. 1. SER versus SNR: 16 × 16 system, 64-QAM symbols.

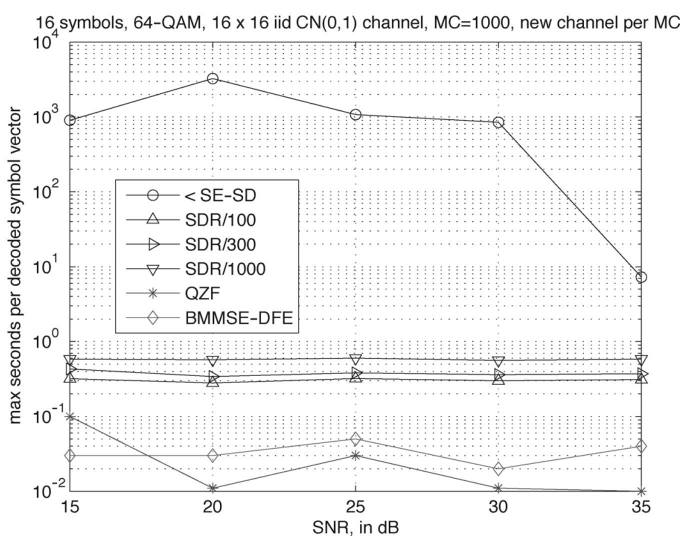

Fig. 2. Worst-case execution time versus SNR: 16 × 16 system, 64-QAM symbols.

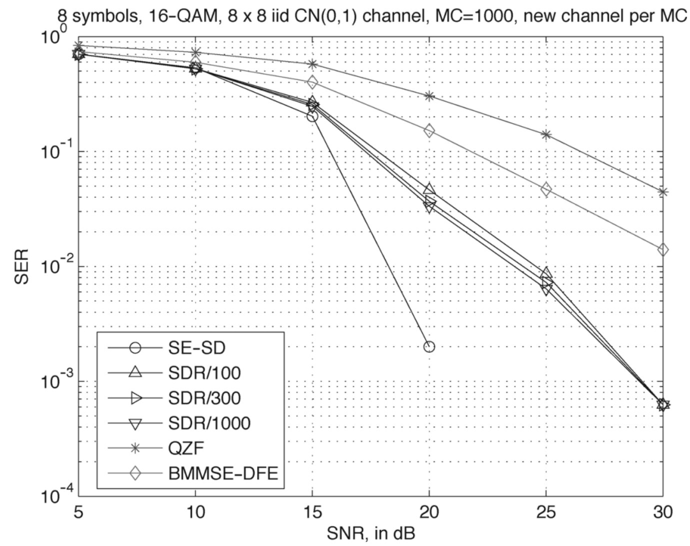

Fig. 3. SER versus SNR: 8 × 8 system, 16-QAM symbols.

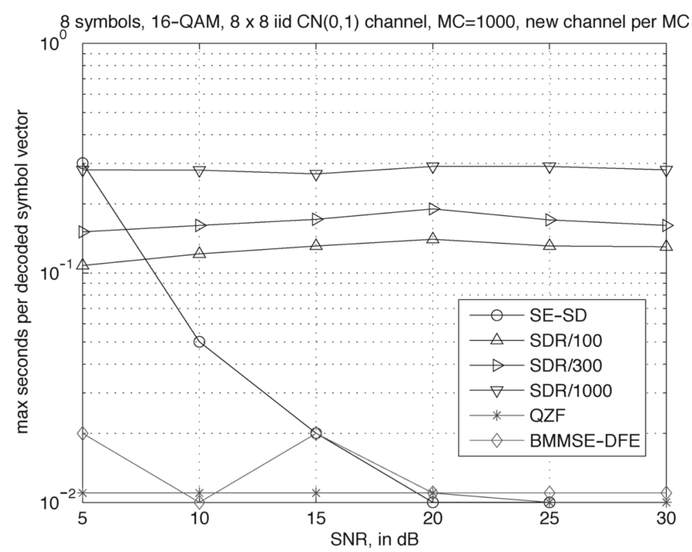

Fig. 4. Worst-case execution time versus SNR: 8 × 8 system, 16-QAM symbols.

metrics were used: symbol error rate (SER) and worst-case execution[2] time. SE-SD was implemented as a Matlab executable (mex) compiled from optimized C code; SDR was implemented using the general-purpose SeDuMi toolbox [10]. As a result, execution time estimates are somewhat biased in favor of SE-SD. The reason for using a measure of worst-case (as opposed to average) complexity is that in online applications, we have to decode within a specified time, and bad channels do happen with positive probability. The choice between execution time or number of floating point operations is debatable, especially because SE-SD was implemented in mex/C; however, we are interested in order-of-magnitude estimates, and differences in execution time are easier to appreciate.

Figs. 1 and 2 show the SER versus SNR and worst-case execution time versus SNR, respectively, for the 16 × 16 system using 64-QAM ($64^{16} \approx 8 \times 10^{28}$). From Fig. 2, it is evident that SE-SD is too complex for this configuration; very long runs are actually not atypical. Due to this, Fig. 2 actually shows a *lower bound* on the worst-case execution time of SE-SD, computed from far fewer realizations. The associated SER cannot be estimated in reasonable time and is therefore not reported in Fig. 1. SDR provides a performance improvement of up to 7.5 dB over BMMSE-DFE. Note that the worst-case complexity of SDR is essentially independent of SNR. In fact, the point-wise complexity of SDR is very stable and predictable for any problem realization. This is good at low to moderate SNR but a drawback at high SNR, where the detection problem becomes easier. Also note that the number of randomization samples used in SDR does not affect the *grosso modo* complexity order, as expected, and a moderate number of randomizations is sufficient.

Figs. 3 and 4 show corresponding results for the 8 × 8 system using 16-QAM ($16^8 \approx 4.3 \times 10^9$). Notice that, in this (far) simpler scenario, SE-SD is much more efficient computationally than SDR, and it always yields the exact ML solution. SDR is

[2]On an Intel Centrino 1.6-GHz system, with 512M RAM.

up to 7.5 dB away from SE-SD, at a uniformly higher computational cost across the range of SNR of interest. It clearly makes no sense to use SDR in this case.

Summarizing, the SD family of detectors exhibits a threshold behavior: it either works very well (for low-enough symbol vector dimension, order of the individual symbol constellation, and high-enough SNR) or it "freezes." The threshold between the two regimes depends on a combination of these three factors. When SD works, it outperforms SDR in terms of complexity and SER performance. In difficult scenarios, SDR offers an attractive alternative relative to earlier solutions.

V. Conclusions

We have proposed a new SDR approach for MIMO detection of high-order QAM constellations. The new approach is the simplest one in the class of SDR detectors for high-order QAM: its worst-case complexity is nearly cubic in the dimension of the transmitted symbol vector and independent of the constellation order for uniform QAM/affine in the constellation order for nonuniform QAM. Under certain conditions, the new approach affords significant improvements in SER over prior methods.

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