Almost-Sure Identifiability of Multidimensional Harmonic Retrieval

Tao Jiang, *Student Member, IEEE*, Nicholas D. Sidiropoulos, *Senior Member, IEEE*, and Jos M. F. ten Berge

***Abstract*—Two-dimensional (2-D) and, more generally, multidimensional harmonic retrieval is of interest in a variety of applications, including transmitter localization and joint time and frequency offset estimation in wireless communications. The associated identifiability problem is key in understanding the fundamental limitations of parametric methods in terms of the number of harmonics that can be resolved for a given sample size. Consider a mixture of 2-D exponentials, each parameterized by amplitude, phase, and decay rate plus frequency in each dimension. Suppose that I equispaced samples are taken along one dimension and, likewise, J along the other dimension. We prove that if the number of exponentials is less than or equal to roughly $IJ/4$, then, assuming sampling at the Nyquist rate or above, the parameterization is almost surely identifiable. This is significant because the best previously known achievable bound was roughly $(I+J)/2$. For example, consider $I = J = 32$; our result yields 256 versus 32 identifiable exponentials. We also generalize the result to N dimensions, proving that the number of exponentials that can be resolved is proportional to total sample size.**

***Index Terms*—Array signal processing, frequency estimation, harmonic analysis, multidimensional signal processing, spectral analysis.**

I. INTRODUCTION

THE PROBLEM of harmonic retrieval is commonly encountered under different disguises in diverse applications in the sciences and engineering [24]. Although one-dimensional (1-D) harmonic retrieval is most common, many applications of multidimensional harmonic retrieval can be found in radar (e.g., [10], [13], and references therein), passive range-angle localization [23], joint 2-D angle and carrier frequency estimation [28], [29], and wireless channel sounding [6]–[9]. In wireless channel sounding, for example, one is interested in jointly estimating several multipath signal parameters like azimuth, elevation, delay, and Doppler, all of which can often be viewed as or transformed into frequency parameters.

A plethora of 1-D as well as multidimensional harmonic retrieval techniques have been developed, ranging from nonparametric Fourier-based methods to modern parametric methods that are not bound by the Fourier resolution limit. In the high signal-to-noise ratio (SNR) regime, parametric methods work well with only a limited number of samples.

One important issue with parametric methods is to determine the maximum number of harmonics that can be resolved for a given total sample size; another is to determine the sample size needed to meet performance specifications.

Identifiability-imposed bounds on sample size are often not the issue in time series analysis because samples are collected along the temporal dimension (hence "inexpensive"), and performance considerations dictate many more samples than what is needed for identifiability. The maximum number of resolvable harmonics comes back into play in situations where data samples along the harmonic mode come at a premium, e.g., in spatial sampling for direction-of-arrival estimation using a uniform linear array (ULA), in which case, one can meet performance requirements with few spatial samples but many temporal samples [25].

Determining the maximum number of resolvable harmonics is a parameter identifiability problem, whose solution for the case of 1-D harmonics goes back to Carathéodory [2]; see also [15] and [26]. In two or higher dimensions, the identifiability problem is considerably harder but also more interesting. The reason is that in many applications of higher dimensional harmonic retrieval, one is constrained in the number of samples that can be taken along certain dimensions, which is usually due to hardware and/or cost limitations. Examples include ultrasound imaging [4] and direction-of-arrival (spatial frequency) estimation. The question that arises is whether the number of samples taken along any particular dimension bounds the overall number of resolvable harmonics or not.

Essentially, all of the work to date on identifiability of multidimensional harmonic retrieval deals with the 2-D case (e.g., [11], [13]) and provides sufficient identifiability conditions that are constrained by $\min(I, J)$, where I denotes the number of samples taken along one dimension, and J denotes the number of samples taken along the other dimension. To the best of our knowledge, the most relaxed condition to date has been derived in [17], which shows that identifiability is determined by the *sum* $I + J$. The result of [17] is deterministic in the sense that no statistical assumptions are needed, aside from the requirement that the frequencies along *each* dimension are distinct. Furthermore, it generalizes naturally to N dimensions for arbitrary N and shows that identifiability improves with increasing N, which is intuitively pleasing. However, the sufficient condition in [17] improves with the *sum* of $I, J, K, \ldots$, whereas total sample size grows with the *product* of $I, J, K, \ldots$. This indicates that significantly stronger results are *possible*.

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T. Jiang and N. D. Sidiropoulos are with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: nikos@ece.umn.edu; jiang@ece.umn.edu).

J. M. F. ten Berge is with the Heijmans Instituut, Rijksuniversiteit Groningen, Groningen, The Netherlands (e-mail: J.M.F.ten.BERGE@ppsw.rug.nl).

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The contribution of this paper is the derivation of stochastic identifiability results for multidimensional harmonic retrieval that fulfill this potential. Our tools allow us to treat the general case of multidimensional complex exponentials that incorporate real exponential components (e.g., decay rates). We thus make no distinction between the terms *harmonic* and *exponential*. We show that if the number of 2-D harmonics is less than or equal to roughly $IJ/4$, then, assuming sampling at the Nyquist rate or above, the parameterization (including the pairing of parameters) is $P_{\mathcal{L}}(\mathbb{C}^{2F})$ almost-surely identifiable, where F is the number of harmonics, and $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex decay/frequency parameters, that is assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$. In plain words, this means that if F is under roughly $IJ/4$, then the model parameters (amplitudes, phases, decay rates, and frequencies, including pairing thereof) that give rise to the observed noiseless data are unique *for almost every selection of complex decay/frequency parameters*, or, if one draws the complex decay/frequency parameters from a continuous distribution over $\mathbb{C}^{2F}$, then the probability that one encounters a nonidentifiable model is zero. This result is subsequently generalized to N dimensions for arbitrary N.

A. Organization

The rest of this paper is structured as follows. We begin with a discussion of notation and preliminaries. Section II summarizes earlier deterministic identifiability results, whereas Section III illuminates the rank properties of the Khatri–Rao matrix product. Both are needed to prove the stochastic identifiability results presented herein. In particular, Section III proves that the Khatri–Rao product is full rank almost surely.[1] Our main contributions are presented in Sections IV and V. Section IV contains the 2-D result, whereas Section V contains its generalization to arbitrary number of dimensions. The proof of the latter is highly technical and is therefore deferred to the Appendix, along with other proofs of auxiliary results. Some comments and extensions of the main results are collected in Section VI. Conclusions are drawn in Section VII.

B. Notation and Some Preliminaries

$\mathbb{C}$ denotes the complex numbers, and $\mathbb{U} = \{x \in \mathbb{C} \,|\, |x| = 1\}$ denotes the unit circle

$$\mathbb{C}^F = \overbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}^{F}$$

and

$$\mathbb{U}^F = \overbrace{\mathbb{U} \times \mathbb{U} \times \cdots \times \mathbb{U}}^{F}.$$

Matrices (vectors) are denoted by boldface capital (lowercase) letters. ^{T} stands for transpose. N denotes the number of dimensions, whereas I_n denotes the number of (equispaced) samples along the nth dimension. An N-dimensional (also known as N-way) array is a dataset that is indexed by N indices $x_{i_1,\ldots,i_N}$, where $i_n \in \{1, \ldots, I_n\}$, and $n = 1, \ldots, N$. We do not follow the usual convention of using i or j to denote $\sqrt{-1}$; instead, we explicitly write $\sqrt{-1}$ when needed and use i (j) as row (respectively, column) index, in accordance with common practice in matrix algebra. We also make extensive use of superscripts to denote variables stemming from a given variable.

The $\texttt{rank}$ of a matrix (two-way array) $\mathbf{A}$ is the smallest number of rank-one matrices needed to decompose $\mathbf{A}$ into a sum of rank-one factors. Each rank-one factor is the outer product of two vectors. Matrix rank can be equivalently defined as the maximum number of linearly independent columns (or rows) that can be drawn from $\mathbf{A}$. We will use $r_{\mathbf{A}}$ to denote the rank of $\mathbf{A}$. The rank of an N-way array is defined as the smallest number of rank-one N-way factors needed to decompose it [12]. Each rank-one N-way factor is the "outer product" of N vectors, meaning that its $(i_1, \ldots, i_N)$th element is given by $a_{f,1,i_1} \cdots a_{f,N,i_N}$, where f is a factor index. Thus, an N-way array of rank F can be written as

$$x_{i_1,\ldots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n,i_n}.$$

The $\texttt{Kruskal-rank}$ or $\texttt{k-rank}$ [12] of a matrix $\mathbf{A}$ (which is denoted by $k_{\mathbf{A}}$) is r if *every* r columns of $\mathbf{A}$ are linearly independent and either $\mathbf{A}$ has r columns or $\mathbf{A}$ contains a set of $r+1$ linearly dependent columns. The k-rank of $\mathbf{A}$ is therefore the maximum number of linearly independent columns that can be drawn from $\mathbf{A}$ in an *arbitrary fashion*. Note that k-rank is generically asymmetric. The k-rank of a matrix need not be equal to the k-rank of its transpose. k-rank is always less than or equal to rank.

A constant-envelope 1-D discrete-time exponential is written as $x_i = ce^{(\sqrt{-1})\omega(i-1)}, i = 1, \ldots, I$, where $c \in \mathbb{C}$ accounts for both amplitude and phase. A nonconstant-envelope 1-D exponential is written as $x_i = ca^{i-1}, i = 1, \ldots, I$, where $a \in \mathbb{C}$ accounts for both decay (or growth) rate and frequency. A 2-D exponential is simply the product of two 1-D exponentials indexed by different independent variables, i.e., $x_{i_1,i_2} = ca_1^{i_1-1}a_2^{i_2-1}$, $i_1 = 1, \ldots, I_1, i_2 = 1, \ldots, I_2$, and likewise for higher dimensions.

An $m \times \rho$ Vandermonde matrix with generators $\alpha_1, \alpha_2, \ldots, \alpha_\rho \in \mathbb{C}$ is given by

$$\mathbf{V} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_\rho \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_\rho^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_\rho^{m-1} \end{bmatrix}.$$

If the generators are distinct, then $\mathbf{V}$ is full rank [24] as well as full k-rank [22]: $k_{\mathbf{V}} = r_{\mathbf{V}} = \min(m, \rho)$.

Let

$$\mathbf{A} = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_i \quad \cdots \quad \mathbf{a}_r]$$
$$\mathbf{B} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_i \quad \cdots \quad \mathbf{b}_r]$$

be two matrices with common number of columns (r). The Khatri–Rao (column-wise Kronecker) matrix product of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \odot \mathbf{B} := [\mathbf{a}_1 \otimes \mathbf{b}_1 \quad \cdots \quad \mathbf{a}_i \otimes \mathbf{b}_i \quad \cdots \quad \mathbf{a}_r \otimes \mathbf{b}_r]$$

[1]This statement has to be interpreted properly; see Section III.

where $\mathbf{a}_i \otimes \mathbf{b}_i$ denotes the Kronecker product of $\mathbf{a}_i$ and $\mathbf{b}_i$.

II. Deterministic Identifiability Results

We will make use of the following results.

Theorem 1: (Identifiability of Low-Rank Decomposition of N-Way Arrays [18], [19]): Consider the F-component N-linear model

$$x_{i_1,\ldots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n,i_n}$$

for $i_n = 1, \ldots, I_n \geq 2$, $n = 1, \ldots, N$, with $c_f \in \mathbb{C}$, $a_{f,n,i_n} \in \mathbb{C}$. Let $\mathbf{A}^{(n)}$ denote the $I_n \times F$ matrix with (i_n, f) element a_{f,n,i_n}. If

$$\sum_{n=1}^{N} k_{\mathbf{A}^{(n)}} \geq 2F + (N-1)$$

then given the N-way array $x_{i_1,\ldots,i_N}$, $i_n = 1, \ldots, I_n$, $n = 1, \ldots, N$, its F rank-one N-way factors

$$c_f \prod_{n=1}^{N} a_{f,n,i_n}, \qquad f = 1, \ldots, F$$

are unique.

Kruskal was the one who developed the backbone result for $N = 3$ and array elements drawn from $\mathbb{R}$ [12]. See also [20]–[22] for other related results.

Theorem 2: (Deterministic Identifiability of N-Dimensional Harmonic Retrieval [17]): Given a sum of F exponentials in N-dimensions

$$x_{i_1,\ldots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n-1}$$

for $i_n = 1, \ldots, I_n \geq 2$, $n = 1, \ldots, N$, with $c_f \in \mathbb{C}$, and $a_{f,n} \in \mathbb{C}$ such that $a_{f_1,n} \neq a_{f_2,n}$, $\forall f_1 \neq f_2$ and all n, if

$$\sum_{n=1}^{N} I_n \geq 2F + (N-1)$$

then there exist unique $(a_{f,n}, n = 1, \ldots, N; c_f)$, $f = 1, \ldots, F$ that give rise to $x_{i_1,\ldots,i_N}$. If an additional M nonexponential dimensions are available

$$x_{i_1,\ldots,i_N,j_1,\ldots,j_M} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n-1} \prod_{m=1}^{M} b_{f,m,j_m} \qquad (1)$$

for $j_m = 1, \ldots, J_m \geq 2$, $m = 1, \ldots, M$, with $b_{f,m,1} = 1$, $\forall f, m$ by convention, then uniqueness (including the associated component vectors along nonexponential dimensions) holds, provided that

$$\sum_{n=1}^{N} I_n + \sum_{m=1}^{M} k_{\mathbf{B}^{(m)}} \geq 2F + (N+M-1)$$

where $\mathbf{B}^{(m)}$ denotes the $J_m \times F$ matrix with (j_m, f) element b_{f,m,j_m}.

III. On Rank and k-Rank of the Khatri–Rao Product

Consider two Vandermonde matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_F \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_F^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{I-1} & \alpha_2^{I-1} & \cdots & \alpha_F^{I-1} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_F \\ \beta_1^2 & \beta_2^2 & \cdots & \beta_F^2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_1^{J-1} & \beta_2^{J-1} & \cdots & \beta_F^{J-1} \end{bmatrix} \qquad (2)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_F$ and $\beta_1, \beta_2, \ldots, \beta_F$ are complex generators. The Khatri–Rao product of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_F \\ \beta_1^2 & \beta_2^2 & \cdots & \beta_F^2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta_1^{J-1} & \beta_2^{J-1} & \cdots & \beta_F^{J-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_F \\ \alpha_1\beta_1 & \alpha_2\beta_2 & \cdots & \alpha_F\beta_F \\ \alpha_1\beta_1^2 & \alpha_2\beta_2^2 & \cdots & \alpha_F\beta_F^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^i\beta_1^j & \alpha_2^i\beta_2^j & \cdots & \alpha_F^i\beta_F^j \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{I-1}\beta_1^{J-1} & \alpha_2^{I-1}\beta_2^{J-1} & \cdots & \alpha_F^{I-1}\beta_F^{J-1} \end{bmatrix}.$$

One can show that full rank (even full k-rank) of both $\mathbf{A}$ and $\mathbf{B}$ does not necessarily guarantee that the Khatri–Rao product $\mathbf{A}\odot\mathbf{B}$ is full rank (let alone full k-rank). For example, let $F = 6$. The generators can be chosen as follows:

$$\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 4, \alpha_5 = 5, \alpha_6 = 6$$
$$\beta_1 = 1, \beta_2 = \sqrt{2}, \beta_3 = \sqrt{3}, \beta_4 = \sqrt{4}, \beta_5 = \sqrt{5}, \beta_6 = \sqrt{6}.$$

With this choice of generators, $\mathbf{A}$ and $\mathbf{B}$ are full k-rank. When $I = 3$ and $J = 2$, the 6×6 Khatri–Rao product $\mathbf{A} \odot \mathbf{B}$ is full rank, hence, full k-rank: $k_{\mathbf{A}\odot\mathbf{B}} = r_{\mathbf{A}\odot\mathbf{B}} = 6$. Now, set $I = 2$ and $J = 3$; the Khatri–Rao product is still 6×6, but[2] its rank is 5.

Irrespective of Vandermonde structure, it is simple to show that

$$r_{\mathbf{A}\odot\mathbf{B}} \leq r_{\mathbf{A}} r_{\mathbf{B}}$$

e.g., by noting that the Khatri–Rao product of $\mathbf{A}$ and $\mathbf{B}$ is a selection of columns drawn from the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$. The rank of the Kronecker product is the product of ranks of the constituent matrices [1].

[2]It will be shown that with proper random sampling, this phenomenon is a measure-zero event; see Theorem 3 and Corollary 1.

The following result provides a deterministic lower bound on the k-rank of the Khatri–Rao product, irrespective of Vandermonde structure. Note that since rank $\geq k$-rank, it also provides a lower bound on rank.

Lemma 1 [22]: Given two matrices $\mathbf{A} \in \mathbb{C}^{I\times F}$ and $\mathbf{B} \in \mathbb{C}^{J\times F}$, if $k_{\mathbf{A}} \geq 1$ and $k_{\mathbf{B}} \geq 1$, then it holds that

$$k_{\mathbf{A}\odot\mathbf{B}} \geq \min(k_{\mathbf{A}} + k_{\mathbf{B}} - 1, F). \quad (3)$$

Other researchers have noted that the Khatri–Rao product appears to exhibit full rank in essentially all cases of practical interest [27], but no rigorous argument has been given to justify this observation to date. The following two results settle this issue.[3]

Theorem 3: For a pair of Vandermonde matrices $\mathbf{A} \in \mathbb{C}^{I\times F}$ and $\mathbf{B} \in \mathbb{C}^{J\times F}$

$$r_{\mathbf{A}\odot\mathbf{B}} = k_{\mathbf{A}\odot\mathbf{B}} = \min(IJ, F), \qquad P_{\mathcal{L}}\left(\mathbb{C}^{2F}\right)\text{-a.s.} \quad (4)$$

where $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex generators for $\mathbf{A}$ and $\mathbf{B}$, assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$.

As an almost direct byproduct, we obtain the following corollary.

Corollary 1: For a pair of matrices $\mathbf{A} \in \mathbb{C}^{I\times F}$ and $\mathbf{B} \in \mathbb{C}^{J\times F}$

$$r_{\mathbf{A}\odot\mathbf{B}} = k_{\mathbf{A}\odot\mathbf{B}} = \min(IJ, F), \quad P_{\mathcal{L}}\left(\mathbb{C}^{(I+J)F}\right)\text{-a.s.} \quad (5)$$

where $P_{\mathcal{L}}(\mathbb{C}^{(I+J)F})$ is the distribution used to draw the $(I+J)F$ complex elements of A and B, which is assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{(I+J)F}$.

Equipped with these results, we proceed to address the main problem of interest herein.

IV. Almost-Sure Identifiability of 2-D Harmonic Retrieval

Proposition 1[4]: Given a sum of F 2-D exponentials

$$x_{i,j} = \sum_{f=1}^{F} c_f a_f^{i-1} b_f^{j-1} \quad (6)$$

for $i = 1, \ldots, I \geq 4$, and $j = 1, \ldots, J \geq 4$, the parameter triples (a_f, b_f, c_f), $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{2F})$ almost-sure unique,[5] where $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex exponential parameters (a_f, b_f), $f = 1, \ldots, F$, which is assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$, provided that there exist four integers I_1, I_2, J_1, J_2 such that

$$I - I_1 - I_2 + \min(I_1 J_1, F) + \min(I_2 J_2, F) \geq 2F \quad (7)$$

subject to

$$\begin{aligned} I_1 + I_2 &\leq I \\ J_1 + J_2 &= J + 1 \\ \min(I_1, I_2, J_1, J_2) &\geq 2. \end{aligned} \quad (8)$$

Proof: We first define a five-way array with typical element

$$\begin{aligned} \hat{x}_{i_1,i_2,i_3,j_1,j_2} &:= x_{i_1+i_2+i_3-2,\, j_1+j_2-1} \\ &= \sum_{f=1}^{F} c_f a_f^{i_1+i_2+i_3-1-1-1} b_f^{j_1+j_2-1-1} \\ &= \sum_{f=1}^{F} c_f a_f^{i_1-1} a_f^{i_2-1} a_f^{i_3-1} b_f^{j_1-1} b_f^{j_2-1} \end{aligned} \quad (9)$$

where $i_\alpha = 1, \ldots, I_\alpha \geq 2$, and $j_\beta = 1, \ldots, J_\beta \geq 2$, for $\alpha = 1, 2, 3$, $\beta = 1, 2$. Since $\min(I, J) \geq 4$ has been assumed in the statement of the proposition, such extension to five ways is always feasible. Define matrices

$$\begin{aligned} \mathbf{A}_\alpha &= \left(a_f^{i_\alpha - 1}\right) \in \mathbb{C}^{I_\alpha \times F} \\ \mathbf{B}_\beta &= \left(b_f^{j_\beta - 1}\right) \in \mathbb{C}^{J_\beta \times F}. \end{aligned} \quad (10)$$

The next step is to nest the five-way array $\hat{x}$ into a three-way array $\overline{x}$ by collapsing two pairs of dimensions as follows:

$$\begin{aligned} \overline{x}_{i_3,k,l} &:= \hat{x}_{\lceil k/J_1\rceil,\, \lceil l/J_2\rceil,\, i_3,\, k-(\lceil k/J_1\rceil-1)J_1,\, l-(\lceil l/J_2\rceil-1)J_2} \\ &= \sum_{f=1}^{F} c_f \Big(a_f^{\lceil k/J_1\rceil-1} a_f^{\lceil l/J_2\rceil-1} a_f^{i_3-1} b_f^{k-(\lceil k/J_1\rceil-1)J_1-1} \\ &\qquad \times b_f^{l-(\lceil l/J_2\rceil-1)J_2-1} \Big) \\ &= \sum_{f=1}^{F} c_f a_f^{i_3-1} \Big(a_f^{\lceil k/J_1\rceil-1} b_f^{k-(\lceil k/J_1\rceil-1)J_1-1} a_f^{\lceil l/J_2\rceil-1} \\ &\qquad \times b_f^{l-(\lceil l/J_2\rceil-1)J_2-1} \Big) \\ &= \sum_{f=1}^{F} c_f a_f^{i_3-1} d_{k,f} e_{l,f} \end{aligned} \quad (11)$$

for $k = 1, \ldots, I_1 J_1$, $l = 1, \ldots, I_2 J_2$, with $d_{k,f}$ and $e_{l,f}$ given by

$$\begin{aligned} d_{k,f} &:= a_f^{\lceil k/J_1\rceil-1} b_f^{k-(\lceil k/J_1\rceil-1)J_1-1} \\ e_{l,f} &:= a_f^{\lceil l/J_2\rceil-1} b_f^{l-(\lceil l/J_2\rceil-1)J_2-1}. \end{aligned} \quad (12)$$

Define matrices

$$\mathbf{D} = (d_{k,f}) \in \mathbb{C}^{I_1 J_1 \times F}, \quad \mathbf{E} = (e_{l,f}) \in \mathbb{C}^{I_2 J_2 \times F}. \quad (13)$$

$\mathbf{D}$ and $\mathbf{E}$ are nothing but

$$\mathbf{D} = \mathbf{A}_1 \odot \mathbf{B}_1, \quad \mathbf{E} = \mathbf{A}_2 \odot \mathbf{B}_2. \quad (14)$$

[3] Proofs can be found in the Appendix.

[4] The result holds true if we switch I and J.

[5] We assume throughout that sampling is at the Nyquist rate or higher to avoid spectral folding. This allows us to restrict attention to discrete-time frequencies in $(-\pi, \pi]$.

Since $\mathbf{A}_3$ is Vandermonde, Theorem 2 can be invoked to claim uniqueness, provided

$$I_3 + k_{\mathbf{D}} + k_{\mathbf{E}} \geq 2F + 3 - 1. \tag{15}$$

Note that for any particular i_3, k, and l, the product $c_f a_f^{i_3-1} d_{k,f} e_{l,f}$ is equal to $c_f a_f^{i-1} b_f^{j-1}$ with the following choice of i and j:

$$\begin{aligned} i &= i_3 + \left\lceil \frac{k}{J_1} \right\rceil + \left\lceil \frac{l}{J_2} \right\rceil - 2 \\ j &= k - \left(\left\lceil \frac{k}{J_1} \right\rceil - 1 \right) J_1 + l - \left(\left\lceil \frac{l}{J_2} \right\rceil - 1 \right) J_2 - 1. \end{aligned}$$

As i_3, k, and l span their range, the corresponding i and j span their respective range. It follows that uniqueness of the F rank-one 3-D factors $c_f a_f^{i_3-1} d_{k,f} e_{l,f}$ is equivalent to uniqueness of the F rank-one 2-D factors $c_f a_f^{i-1} b_f^{j-1}$, $f = 1, \ldots, F$. Therefore, the rank-one factors $c_f a_f^{i-1} b_f^{j-1}$ and, hence, the triples (a_f, b_f, c_f), $f = 1, \ldots, F$ are unique, provided that (15) holds true. Invoking Theorem 3, almost-sure uniqueness holds, provided there exist integers $I_1, I_2, I_3, J_1, J_2 \geq 2$ such that

$$I_3 + \min(I_1 J_1, F) + \min(I_2 J_2, F) \geq 2F + 2 \tag{16}$$

subject to[6]

$$\begin{cases} I_1 + I_2 + I_3 = I + 2 \\ J_1 + J_2 = J + 1 \\ \min(I_1, I_2, I_3, J_1, J_2) \geq 2. \end{cases} \tag{17}$$

Setting $I_3 = I + 2 - I_1 - I_2$, we obtain

$$I - I_1 - I_2 + \min(I_1 J_1, F) + \min(I_2 J_2, F) \geq 2F$$

subject to

$$\begin{cases} I_1 + I_2 \leq I \\ J_1 + J_2 = J + 1 \\ \min(I_1, I_2, J_1, J_2) \geq 2 \end{cases}$$

and the proof is complete. □

Theorem 4[7]: Given a sum of F 2-D exponentials

$$x_{i,j} = \sum_{f=1}^{F} c_f a_f^{i-1} b_f^{j-1} \tag{18}$$

for $i = 1, \ldots, I \geq 4$, and $j = 1, \ldots, J \geq 4$, the parameter triples (a_f, b_f, c_f), $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{2F})$ almost-sure unique, where $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex exponential parameters (a_f, b_f), $f = 1, \ldots, F$, which is assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$, provided that

$$F \leq \left\lfloor \frac{I}{2} \right\rfloor \left\lceil \frac{J}{2} \right\rceil. \tag{19}$$

[6]The first two conditions assure that we do not index beyond the available data sample.

[7]The Theorem holds true if I and J are switched.

Proof: If both I and J are even numbers , pick $I_1 = I_2 = I/2$, $J_1 = J/2$, and $J_2 = (J+2)/2$ [thereby satisfying (8)], and (7) becomes

$$\min\left(\frac{IJ}{4}, F\right) + \min\left(\frac{I(J+2)}{4}, F\right) \geq 2F \tag{20}$$

which is satisfied for any $F \leq IJ/4$. If I is even and J is odd, pick $I_1 = I_2 = I/2$, and $J_1 = J_2 = (J+1)/2$ [thereby satisfying (8)], and (7) becomes

$$\min\left(\frac{I(J+1)}{4}, F\right) + \min\left(\frac{I(J+1)}{4}, F\right) \geq 2F \tag{21}$$

which is satisfied for any $F \leq I(J+1)/4$. If both I and J are odd, pick $I_1 = (I-1)/2$, $I_2 = (I+1)/2$, $J_1 = J_2 = (J+1)/2$ [satisfying (8)], and (7) becomes

$$\min\left(\frac{(I-1)(J+1)}{4}, F\right) + \min\left(\frac{(I+1)(J+1)}{4}, F\right) \geq 2F \tag{22}$$

satisfied for any $F \leq (I-1)(J+1)/4$. Finally, if I is odd and J is even, pick $I_1 = (I-1)/2$, $I_2 = (I+1)/2$, $J_1 = J/2$, and $J_2 = (J+2)/2$ [satisfying (8)], and (7) becomes

$$\min\left(\frac{(I-1)(J)}{4}, F\right) + \min\left(\frac{(I+1)(J+2)}{4}, F\right) \geq 2F \tag{23}$$

satisfied for any $F \leq (I-1)J/4$. Invoking Proposition 1 completes the proof. □

Remark 1: Some reflection reveals that the argument behind the proof of Theorem 4 (and its N-D generalization: Theorem 5) is in fact constructive, leading to an eigenvalue solution that recovers everything *exactly* under only the model identifiability condition in the Theorem, *in the noiseless case*. Matlab code can be found at http://www.ece.umn.edu/users/nikos/public_html/3SPICE/code.html.

Remark 2: It is interesting to note that equations-versus-unknowns considerations indicate a bound of $IJ/3$, without taking the pairing issue into consideration. To see this, note that each of the F 2-D exponential components is parameterized by three complex parameters, and a total of IJ complex data points are given. If the equations-versus-unknowns bound is violated, then, under certain conditions, the implicit function theorem indicates that infinitely many ambiguous solutions exist in the neighborhood of the true solution.

V. N-DIMENSIONAL CASE

The result can be generalized to the N-dimensional case. Although the spirit of the associated proof is clear, the mathematical argument is highly technical. This is so primarily because one is forced to use a recursive dimensionality-embedding argument to preserve generality. We therefore state the result and defer the proof to the end of the Appendix, noting that Figs. 1 and 2 help convey the essence of the proof to the interested reader.

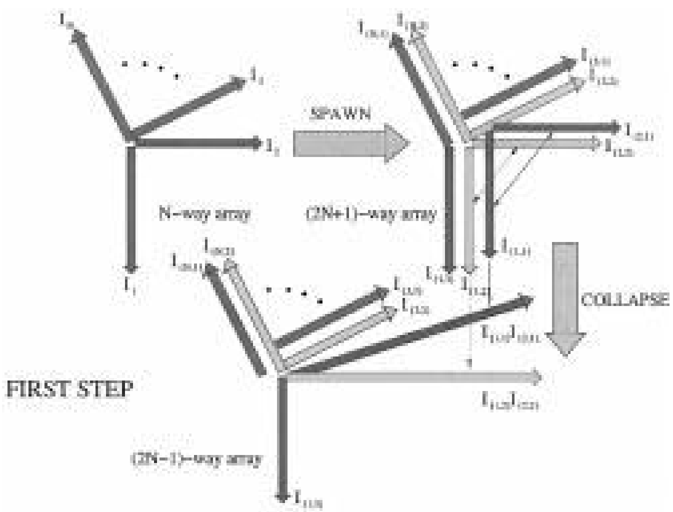

Fig. 1. First step in the proof of the N-dimensional case.

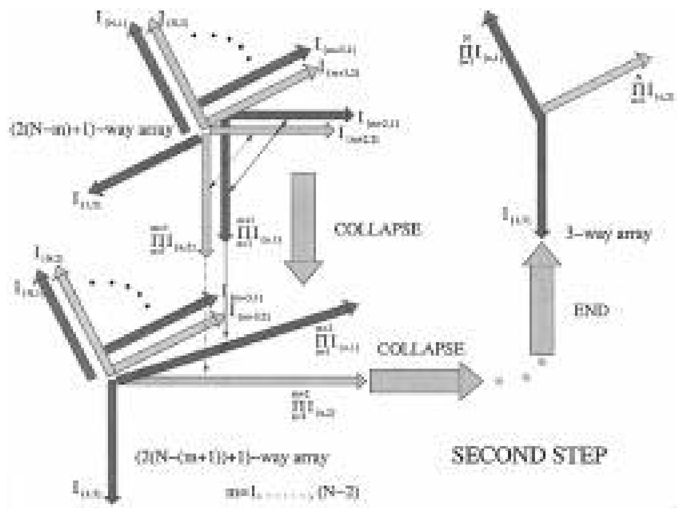

Fig. 2. Second step in the proof of the N-dimensional case.

Theorem 5[8]*:* Given a sum of F N-D exponentials

$$x_{i_1,\,\ldots,\,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,\,n}^{i_n - 1} \tag{24}$$

for $i_n = 1, \ldots, I_n \geq 4$, $n = 1, \ldots, N$, the parameter $(N+1)$-tuples $(a_{f,1}, \ldots, a_{f,N}, c_f)$, $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{NF})$ almost-sure unique, where $P_{\mathcal{L}}(\mathbb{C}^{NF})$ is the distribution used to draw the NF complex exponential parameters $(a_{f,1}, \ldots, a_{f,N})$, for $f = 1, \ldots, F$, which is assumed continuous with respect to Lebesgue measure in $\mathbb{C}^{NF}$, provided that

$$F \leq \left\lfloor \frac{I_1}{2} \right\rfloor \prod_{n=2}^{N} \left\lceil \frac{I_n}{2} \right\rceil. \tag{25}$$

VI. Comments and Extensions

The restriction of at least four samples per dimension is an artifact of the proof. In fact, we can also treat cases with less than four samples in any dimension(s). However, in the 2-D case with less than four samples per dimension, our approach does not yield anything significant. In the N-D case, having less than four samples along certain dimensions breaks the symmetry of the problem, forcing us to separately consider cases, depending on the number and sample size distribution of dimensions having less than four samples. This prohibits a concise unifying treatment. Nevertheless, individual cases can be easily dealt with, given the tools developed herein.

A. Constant-Envelope Exponentials

So far, we have considered multidimensional complex exponentials that incorporate real exponential components. In many applications, one deals with constant-envelope complex exponentials. The proof of Theorem 4 carries through verbatim in this case, except that one needs to ensure that Theorem 3 holds for generators drawn from the unit circle $\mathbb{U}$. This is easy, because the generic example that shows that the determinant is nontrivial in the proof of Theorem 3 was actually constructed using generators drawn from the unit circle. We therefore have the following Corollary.

Corollary 2: Given a sum of F 2-D constant-envelope complex exponentials

$$x_{i,\,j} = \sum_{f=1}^{F} c_f e^{\sqrt{-1}\omega_f (i-1)} e^{\sqrt{-1}\upsilon_f (j-1)}$$

for $i = 1, \ldots, I \geq 4$, and $j = 1, \ldots, J \geq 4$, the parameter triples $(e^{\sqrt{-1}\omega_f}, e^{\sqrt{-1}\upsilon_f}, c_f)$, $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\mathbb{U}^{2F})$ almost-surely unique, provided that

$$F \leq \left\lfloor \frac{I}{2} \right\rfloor \left\lceil \frac{J}{2} \right\rceil.$$

The same argument holds for Proposition 4 and Theorem 5 in the case of constant-envelope complex exponentials; we skip the corresponding statements for brevity.

B. Common Frequency Mode

In most applications, having two or more *identical* frequencies along a certain dimension is a measure zero event. Having two frequencies close to each other is very common, but this affects performance, rather than identifiability. In certain applications, identical frequencies along one or two dimensions are, in fact, a modeling assumption, motivated by proximity of actual frequencies and compactness of model parameterization [13]. For this reason, it is of interest to investigate identifiability subject to common frequency constraints. This can be handled using the tools developed herein, but one needs to check on a case-by-case basis, depending on the "common mode configuration."

- How many distinct frequencies ("batches") per dimension?
- How many components per batch?
- What is the pairing across dimensions?

[8]The Theorem holds true for any permutation of $\{I_n\}_{n=1}^{N}$.

In general, the problem is combinatorial, and a unified treatment does not seem to be possible. The reason is that one needs to construct a "generic" example (cf. the proof of Theorem 3) to demonstrate that the determinant of the associated Khatri–Rao product is nontrivial for each common mode configuration. We illustrate how this situation can be handled in the 2-D case with a pair of 2-D exponentials having one frequency in common. Interestingly, we obtain exactly the same identifiability condition as before. The proof of the following result can be found in the Appendix.

Proposition 2: Given a sum of F 2-D exponentials

$$x_{i,j} = \sum_{f=1}^{F} c_f a_f^{i-1} b_f^{j-1}$$

for $i = 1, \ldots, I \geq 4$, and $j = 1, \ldots, J \geq 4$, with $b_2 = b_1$, the parameter triples (a_f, b_f, c_f), $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{2F-1})$ almost-surely unique, where $P_{\mathcal{L}}(\mathbb{C}^{2F-1})$ is the distribution used to draw the $(2F-1)$ complex exponential parameters $(a_1, a_2, \ldots, a_F, b_1, b_3, \ldots, b_F)$, which is assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F-1}$, provided that

$$F \leq \left\lfloor \frac{I}{2} \right\rfloor \left\lceil \frac{J}{2} \right\rceil.$$

C. Nonexponential Dimension(s)

In certain situations, the signals along one dimension are not exponentials, e.g., in uniform rectangular sensor array processing with two exponential (spatial) dimensions and a nonexponential temporal dimension. Our results can be extended to handle this case as well. As an example, we have the following result.[9]

Proposition 3: Consider

$$x_{i,j,k} = \sum_{f=1}^{F} c_f a_f^{i-1} b_f^{j-1} s_{k,f}$$

for $i = 1, \ldots, I$, and $j = 1, \ldots, J$, where $k = 1, \ldots, K$ is a temporal index, and assume that the temporal signal matrix $\mathbf{S} = (s_{k,f}) \in \mathbb{C}^{K \times F}$ is full column rank F. If $\max(I, J) \geq 3$ and

$$F \leq IJ - \min(I, J)$$

then the parameterization in terms of $(a_f, b_f, c_f, \{s_{k,f}\}_{k=1}^{K})$, $f = 1, \ldots, F$ is $P_{\mathcal{L}}(\mathbb{C}^{2F})$ almost-surely unique, where $P_{\mathcal{L}}(\mathbb{C}^{2F})$ is the distribution used to draw the $2F$ complex exponential parameters (a_f, b_f), $f = 1, \ldots, F$, which is assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$.

[9]Note that, assuming sufficiently many temporal samples and persistence of excitation, and taking $M_3 = L_3 = L = 1$ in of [6, Eq. (22)], yields $F \leq \min(I(J-1), J(I-1)) = IJ - \max(I, J)$; this is worse but close to our result in Proposition 3, albeit [6] contains no proof.

VII. Conclusions

We have derived stochastic identifiability results for multidimensional harmonic retrieval. The sufficient conditions provided are the most relaxed to date. The sufficient condition for the 2-D case is not far from equations-versus-unknowns considerations—hence additional improvements, if any, will be marginal. In the N-D case, the resolvability bound is proportional to total sample size, but the proportionality factor is dependent on N. Although this is not a serious limitation, it does indicate that one moves further from the equations-versus-unknowns bound in higher dimensions. It remains to be seen whether a significantly tighter bound can be found in higher dimensions.

Appendix

We will need to invoke the following Lemma.

Lemma 2: Consider an analytic function $h(\mathbf{x})$ of several complex variables $\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{C}^n$. If h is nontrivial in the sense that there exists $\mathbf{x}_0 \in \mathbb{C}^n$ such that $h(\mathbf{x}_0) \neq 0$, then the zero set of $h(\mathbf{x})$

$$\mathcal{Z} := \{\mathbf{x} \in \mathbb{C}^n | h(\mathbf{x}) = 0\}$$

is of measure (Lebesgue measure in $\mathbb{C}^n$) zero.

This Lemma is known (e.g., [26]), but we have not been able to find a satisfactory proof in the literature. We therefore include a simple proof for completeness.

Proof of Lemma 2: If $n = 1$, it is well known that $\mathcal{Z}$ is countable (e.g., see [3, Th. 3.7][10]). For $n > 1$, define $g(\mathbf{x}) = 1$ if $h(\mathbf{x}) = 0$, and $g(\mathbf{x}) = 0$ otherwise. The measure of $\mathcal{Z}$ is the integral of $g(\mathbf{x})$ over $\mathbb{C}^n$. Fix $x_2, x_3, \ldots, x_n$, and consider the single-variable function $h(x_1, x_2, \ldots, x_n)$. This is analytic in x_1; hence, its zero set is of measure zero. This means that for any fixed $x_2, \ldots, x_n$

$$\int g(x_1, x_2, \ldots, x_n)\, dx_1 = 0.$$

Hence

$$\begin{aligned}\int \cdots \int g\, dx_1\, dx_2 \cdots dx_n &= \int \cdots \left(\int g\, dx_1\right) dx_2 \cdots dx_n \\ &= \int \cdots \int 0\, dx_2 \cdots dx_n \\ &= 0.\end{aligned}$$

Note that the argument works, irrespective of order of integration—hence, the multidimensional integral is indeed zero by Fubini's theorem. This completes the proof. □

Proof of Theorem 3: We will show that

$$r_{\mathbf{A} \odot \mathbf{B}} = k_{\mathbf{A} \odot \mathbf{B}} = \min(IJ, F), \quad P_{\mathcal{L}}\left(\mathbb{C}^{2F}\right) \text{ a.s.}$$

The general case can be reduced to the $IJ = F$ case. If $IJ \leq F$, it suffices to prove that the result holds for an *arbitrary selection* of IJ columns; if $IJ \geq F$, then it suffices to prove that the result holds for any row-reduced square submatrix.

[10]Any uncountable set in the complex plane must have at least one limit point because any complex Cauchy sequence must have one and only one complex limit.

When $IJ = F$, full rank and full k-rank can be established by showing that the determinant of $\mathbf{A} \odot \mathbf{B}$ is nonzero. Define

$$H(\alpha_1, \ldots, \alpha_F, \beta_1, \ldots, \beta_F) = \det(\mathbf{A}(\alpha_1, \ldots, \alpha_F) \odot \mathbf{B}(\beta_1, \ldots, \beta_F)).$$

H is a polynomial in several variables and hence is analytic. In order to establish the desired result, it suffices to show that H is nontrivial. This requires a "generic" example that works for any I, J, F. This can be constructed as follows. For any given I, J, F with $2 \le I \le F$ and $2 \le J \le F$, $IJ = F$, define the generators $\alpha_f = e^{\sqrt{-1}(2\pi/F)J(f-1)}$ and $\beta_f = e^{\sqrt{-1}(2\pi/F)(f-1)}$ for $f = 1, \ldots, F$. It can be verified that with this choice of generators for $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \odot \mathbf{B}$ is itself a Vandermonde matrix with generators $(1, e^{\sqrt{-1}(2\pi/F)}, \ldots, e^{\sqrt{-1}(2\pi/F)(F-1)})$ and, therefore, full rank. This shows that $H(\alpha_1, \ldots, \alpha_F, \beta_1, \ldots, \beta_F)$ is a nontrivial polynomial in $\mathbb{C}^{2F}$. Invoking the analytic function Lemma 2, $H(\alpha_1, \ldots, \alpha_F, \beta_1, \ldots, \beta_F)$ is nonzero almost everywhere, except for a measure zero subset of $\mathbb{C}^{2F}$. □

Remark 3: An alternative proof of Theorem 3 can be constructed by using the theory of Lagrange interpolation in several variables [5], [14], [16]. The advantage of such an approach is that it affords geometric insight that facilitates the construction of full-rank examples and counter-examples. The disadvantage is that the proof requires a long and delicate argument.

Proof of Corollary 1: It is again sufficient to consider the case $IJ = F$. The generic example provided for a pair of Vandermonde matrices can also be used here to show that the determinant of the square Khatri–Rao product of two matrices of appropriate dimensions (but otherwise arbitrary) is a nontrivial polynomial in $(I + J)F$ complex variables; therefore, the analytic function Lemma 2 applies. □

We will need the following preparatory results to prove Theorem 5.

Proposition 4: Given N Vandermonde matrices $\mathbf{A}_n \in \mathbb{C}^{I_n \times F}$ for $n = 1, \ldots, N \ge 2$

$$r_{\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N} = k_{\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N} = \min\left(\prod_{n=1}^{N} I_n, F\right), \quad P_{\mathcal{L}}\left(\mathbb{C}^{NF}\right)\text{-a.s.} \tag{26}$$

where $P_{\mathcal{L}}(\mathbb{C}^{NF})$ is the distribution used to draw the NF complex generators for $\mathbf{A}_n$, $n = 1, \ldots, N$, which is assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{NF}$.

Proof: The general case can be reduced to the $\prod_{n=1}^{N} I_n = F$ case. When $\prod_{n=1}^{N} I_n = F$, the full rank and full k-rank of $(\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N)$ is equivalent to its determinant being nonzero. Define

$$H(\alpha_{1,1}, \ldots, \alpha_{1,F}, \ldots, \alpha_{N,1}, \ldots, \alpha_{N,F}) = \det(\mathbf{A}_1(\alpha_{1,1}, \ldots, \alpha_{1,F}) \odot \cdots \odot \mathbf{A}_N(\alpha_{N,1}, \ldots, \alpha_{N,F}))$$

where $\alpha_{n,f}$ is the fth generator of $\mathbf{A}_n$, $n = 1, \ldots, N$, and $f = 1, \ldots, F$. H is a polynomial in NF variables and, hence, is analytic in $\mathbb{C}^{NF}$. It therefore suffices to show that H is nontrivial. The following generic example works for any F and $I_n \ge 2$, $n = 1, \ldots, N$, showing that H is nontrivial

$$\alpha_{n,f} = \exp\left(\left(\sqrt{-1}\right)(2\pi/F)\left(\prod_{k=1}^{n-1} I_k\right)(f-1)\right)$$

for $n = 1, \ldots, N$, $f = 1, \ldots, F$. It can be verified that with this choice of generators for $\mathbf{A}_n$, $n = 1, \ldots, N$, $\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N$ is a Vandermonde matrix with generators $(1, e^{(\sqrt{-1})(2\pi/F)}, \ldots, e^{(\sqrt{-1})(2\pi/F)(F-1)})$ and is, therefore, full rank. □

Proposition 5[11]*:* Given a sum of F N-D exponentials

$$x_{i_1, \ldots, i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n - 1} \tag{27}$$

for $i_n = 1, \ldots, I_n \ge 4$, $n = 1, \ldots, N$, the parameter $(N+1)$-tuples $(a_{f,1}, \ldots, a_{f,N}, c_f)$, $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{NF})$ almost-surely unique, where $P_{\mathcal{L}}(\mathbb{C}^{NF})$ is the distribution used to draw the NF complex exponential parameters $(a_{f,1}, \ldots, a_{f,N})$ for $f = 1, \ldots, F$, which is assumed continuous with respect to the Lebesgue measure in $\mathbb{C}^{NF}$, provided that there exist $2N$ integers $I_{n,j}$ for $n = 1, \ldots, N$, $j = 1, 2$ such that

$$I_1 - I_{1,1} - I_{1,2} + \min\left(\prod_{n=1}^{N} I_{n,1}, F\right) + \min\left(\prod_{n=1}^{N} I_{n,2}, F\right) \ge 2F \tag{28}$$

subject to

$$\begin{cases} I_{1,1} + I_{1,2} \le I_1 \\ I_{n,1} + I_{n,2} = I_n + 1, & n = 2, \ldots, N \\ I_{n,j} \ge 2, & \forall n, j. \end{cases} \tag{29}$$

Proof: We first extend the given N-way array to a $(2N+1)$-way array with typical element

$$\begin{aligned} \hat{x}_{i_{1,1}, i_{1,2}, i_{1,3}, i_{2,1}, i_{2,2}, \ldots, i_{N,1}, i_{N,2}} &:= x_{i_{1,1}+i_{1,2}+i_{1,3}-2,\, i_{2,1}+i_{2,2}-1, \ldots, i_{N,1}+i_{N,2}-1} \\ &= \sum_{f=1}^{F} c_f \left(a_{f,1}^{i_{1,1}+i_{1,2}+i_{1,3}-3} \prod_{n=2}^{N} a_{f,n}^{i_{n,1}+i_{n,2}-2} \right) \\ &= \sum_{f=1}^{F} c_f \left(a_{f,1}^{i_{1,3}-1} \prod_{n=1}^{N} a_{f,n}^{i_{n,1}+i_{n,2}-2} \right) \\ &= \sum_{f=1}^{F} c_f \left(a_{f,1}^{i_{1,3}-1} \prod_{n=1}^{N} a_{f,n}^{i_{n,1}-1} \prod_{n=1}^{N} a_{f,n}^{i_{n,2}-1} \right) \end{aligned} \tag{30}$$

where

$$\begin{aligned} i_{n,j} &= 1, \ldots, I_{n,j} \ge 2 \\ i_{1,3} &= 1, \ldots, I_{1,3} \ge 2 \\ n &= 1, \ldots, N, \; j = 1, 2. \end{aligned}$$

[11]The Proposition holds true for any permutation of $\{I_n\}_{n=1}^{N}$.

Such extension is always possible under our working assumption that $I_n \geq 4, \forall n$. We also need the following constraints to avoid indexing beyond the available data sample:

$$\begin{cases} I_{1,1}+I_{1,2}+I_{1,3}=I_1+2 \\ I_{n,1}+I_{n,2}=I_n+1,\ n=2,\ldots,N. \end{cases} \tag{31}$$

Define matrices

$$\mathbf{A}_{n,j}=\left(a_{f,n}^{i_{n,j}-1}\right)\in\mathbb{C}^{I_{n,j}\times F}$$
$$\mathbf{A}_{1,3}=\left(a_{f,1}^{i_{1,3}-1}\right)\in\mathbb{C}^{I_{1,3}\times F} \tag{32}$$

for $n=1,\ldots,N$, $j=1,2$. Next, we compress the $(2N+1)$-way $\hat{x}$ array into a three-way array $\overline{x}$. We do this in two steps for clarity. The first step is to nest $\hat{x}$ into a $(2N-1)$-way array $\hat{x}^{(1)}$. This process is illustrated in Fig. 1.

$$\begin{aligned}
&\hat{x}^{(1)}_{i_{1,3},k^{(1)},l^{(1)},i_{3,1},i_{3,2},\ldots,i_{N,1},i_{N,2}}\\
&\quad=\hat{x}_{\lceil k^{(1)}/I_{2,1}\rceil,\lceil l^{(1)}/I_{2,2}\rceil,i_{1,3},k^{(1)}-(\lceil k^{(1)}/I_{2,1}\rceil-1)I_{2,1}}\\
&\qquad\qquad {}_{l^{(1)}-(\lceil l^{(1)}/I_{2,2}\rceil-1)I_{2,2},i_{3,1},i_{3,2},\ldots,i_{N,1},i_{N,2}}\\
&\quad=\sum_{f=1}^{F}c_f\left(a_{f,1}^{\lceil k^{(1)}/I_{2,1}\rceil-1}a_{f,1}^{\lceil l^{(1)}/I_{2,2}\rceil-1}a_{f,1}^{i_{1,3}-1}\right.\\
&\qquad\times a_{f,2}^{k^{(1)}-(\lceil k^{(1)}/I_{2,1}\rceil-1)I_{2,1}-1}a_{f,2}^{l^{(1)}-(\lceil l^{(1)}/I_{2,2}\rceil-1)I_{2,2}-1}\\
&\qquad\left.\times\prod_{n=3}^{N}a_{f,n}^{i_{n,1}-1}\prod_{n=3}^{N}a_{f,n}^{i_{n,2}-1}\right)\\
&\quad=\sum_{f=1}^{F}c_f\left(a_{f,1}^{i_{1,3}-1}a_{f,1}^{\lceil k^{(1)}/I_{2,1}\rceil-1}a_{f,2}^{k^{(1)}-(\lceil k^{(1)}/I_{2,1}\rceil-1)I_{2,1}-1}\right.\\
&\qquad\times a_{f,1}^{\lceil l^{(1)}/I_{2,2}\rceil-1}a_{f,2}^{l^{(1)}-(\lceil l^{(1)}/I_{2,2}\rceil-1)I_{2,2}-1}\\
&\qquad\left.\times\prod_{n=3}^{N}a_{f,n}^{i_{n,1}-1}\prod_{n=3}^{N}a_{f,n}^{i_{n,2}-1}\right)\\
&\quad=\sum_{f=1}^{F}c_f a_{f,1}^{i_{1,3}-1}d^{(1)}_{k^{(1)},f}e^{(1)}_{l^{(1)},f}\prod_{n=3}^{N}a_{f,n}^{i_{n,1}-1}\prod_{n=3}^{N}a_{f,n}^{i_{n,2}-1}
\end{aligned} \tag{33}$$

for $k^{(1)}=1,\ldots,I_{1,1}I_{2,1}$, $l^{(1)}=1,\ldots,I_{1,2}I_{2,2}$, with $d^{(1)}_{k^{(1)},f}$ and $e^{(1)}_{l^{(1)},f}$ given by

$$\begin{aligned}
d^{(1)}_{k^{(1)},f}&:=a_{f,1}^{\lceil k^{(1)}/I_{2,1}\rceil-1}a_{f,2}^{k^{(1)}-(\lceil k^{(1)}/I_{2,1}\rceil-1)I_{2,1}-1}\\
e^{(1)}_{l^{(1)},f}&:=a_{f,1}^{\lceil l^{(1)}/I_{2,2}\rceil-1}a_{f,2}^{l^{(1)}-(\lceil l^{(1)}/I_{2,2}\rceil-1)I_{2,2}-1}.
\end{aligned} \tag{34}$$

Define matrices

$$\mathbf{D}^{(1)}=\left(d^{(1)}_{k^{(1)},f}\right)\in\mathbb{C}^{I_{1,1}I_{2,1}\times F}$$
$$\mathbf{E}^{(1)}=\left(e^{(1)}_{l^{(1)},f}\right)\in\mathbb{C}^{I_{1,2}I_{2,2}\times F} \tag{35}$$

and note that

$$\mathbf{D}^{(1)}=\mathbf{A}_{1,1}\odot\mathbf{A}_{2,1},\quad \mathbf{E}^{(1)}=\mathbf{A}_{1,2}\odot\mathbf{A}_{2,2}. \tag{36}$$

The next step is to show that starting from $m=1$, we can recursively nest the $(2(N-m)+1)$-way array $\hat{x}^{(m)}$ into a $(2(N-(m+1))+1)$-way array $\hat{x}^{(m+1)}$, $m=1,\ldots,(N-2)$. This step is illustrated in Fig. 2.

$$\begin{aligned}
&\hat{x}^{(m+1)}_{i_{1,3},k^{(m+1)},l^{(m+1)},i_{(m+3),1},i_{(m+3),2},\ldots,i_{N,1},i_{N,2}}\\
&\quad=\hat{x}^{(m)}_{i_{1,3},\lceil k^{(m+1)}/I_{(m+2),1}\rceil,\lceil l^{(m+1)}/I_{(m+2),2}\rceil}\\
&\qquad\quad {}_{k^{(m+1)}-(\lceil k^{(m+1)}/I_{(m+2),1}\rceil-1)I_{(m+2),1}}\\
&\qquad\quad {}_{l^{(m+1)}-(\lceil l^{(m+1)}/I_{(m+2),2}\rceil-1)I_{(m+2),2}}\\
&\qquad\quad {}_{i_{(m+3),1},i_{(m+3),2},\ldots,i_{N,1},i_{N,2}}\\
&\quad=\sum_{f=1}^{F}c_f\left(a_{f,1}^{i_{1,3}-1}d^{(m)}_{\lceil k^{(m+1)}/I_{(m+2),1}\rceil,f}e^{(m)}_{\lceil l^{(m+1)}/I_{(m+2),2}\rceil,f}\right.\\
&\qquad\times a_{f,(m+2)}^{k^{(m+1)}-(\lceil k^{(m+1)}/I_{(m+2),1}\rceil-1)I_{(m+2),1}-1}\\
&\qquad\times a_{f,(m+2)}^{l^{(m+1)}-(\lceil l^{(m+1)}/I_{(m+2),2}\rceil-1)I_{(m+2),2}-1}\\
&\qquad\left.\times\prod_{n=(m+3)}^{N}a_{f,n}^{i_{n,1}-1}\prod_{n=(m+3)}^{N}a_{f,n}^{i_{n,2}-1}\right)\\
&\quad=\sum_{f=1}^{F}c_f\left(a_{f,1}^{i_{1,3}-1}d^{(m)}_{\lceil k^{(m+1)}/I_{(m+2),1}\rceil,f}\right.\\
&\qquad\times a_{f,(m+2)}^{k^{(m+1)}-(\lceil k^{(m+1)}/I_{(m+2),1}\rceil-1)I_{(m+2),1}-1}\\
&\qquad\times e^{(m)}_{\lceil l^{(m+1)}/I_{(m+2),2}\rceil,f}\\
&\qquad\times a_{f,(m+2)}^{l^{(m+1)}-(\lceil l^{(m+1)}/I_{(m+2),2}\rceil-1)I_{(m+2),2}-1}\\
&\qquad\left.\times\prod_{n=(m+3)}^{N}a_{f,n}^{i_{n,1}-1}\prod_{n=(m+3)}^{N}a_{f,n}^{i_{n,2}-1}\right)\\
&\quad=\sum_{f=1}^{F}c_f\left(a_{f,1}^{i_{1,3}-1}d^{(m+1)}_{k^{(m+1)},f}e^{(m+1)}_{l^{(m+1)},f}\right.\\
&\qquad\left.\times\prod_{n=(m+3)}^{N}a_{f,n}^{i_{n,1}-1}\prod_{n=(m+3)}^{N}a_{f,n}^{i_{n,2}-1}\right)
\end{aligned} \tag{37}$$

for

$$k^{(m+1)}=1,\ldots,\prod_{n=1}^{(m+2)}I_{n,1}$$
$$l^{(m+1)}=1,\ldots,\prod_{n=1}^{(m+2)}I_{n,2}$$

with $d^{(m+1)}_{k^{(m+1)},f}$ and $e^{(m+1)}_{l^{(m+1)},f}$ given by

$$\begin{aligned}
d^{(m+1)}_{k^{(m+1)},f}&:=d^{(m)}_{\lceil k^{(m+1)}/I_{(m+2),1}\rceil,f}\\
&\qquad\times a_{f,(m+2)}^{k^{(m+1)}-(\lceil k^{(m+1)}/I_{(m+2),1}\rceil-1)I_{(m+2),1}-1}\\
e^{(m+1)}_{l^{(m+1)},f}&:=e^{(m)}_{\lceil l^{(m+1)}/I_{(m+2),2}\rceil,f}\\
&\qquad\times a_{f,(m+2)}^{l^{(m+1)}-(\lceil l^{(m+1)}/I_{(m+2),2}\rceil-1)I_{(m+2),2}-1}.
\end{aligned} \tag{38}$$

Define matrices

$$\mathbf{D}^{(m+1)} = \left(d^{(m+1)}_{k^{(m+1)},\, f}\right) \in \mathbb{C}^{\prod_{n=1}^{(m+2)} I_{n,\,1}\times F}$$
$$\mathbf{E}^{(m+1)} = \left(e^{(m+1)}_{l^{(m+1)},\, f}\right) \in \mathbb{C}^{\prod_{n=1}^{(m+2)} I_{n,\,2}\times F}. \quad (39)$$

$\mathbf{D}^{(m+1)}$ and $\mathbf{E}^{(m+1)}$ can be written as

$$\begin{aligned}\mathbf{D}^{(m+1)} &= \mathbf{D}^{(m)} \odot \mathbf{A}_{(m+2),\,1}\\ &= \mathbf{A}_{1,\,1} \odot \cdots \odot \mathbf{A}_{(m+1),\,1} \odot \mathbf{A}_{(m+2),\,1}\\ \mathbf{E}^{(m+1)} &= \mathbf{E}^{(m)} \odot \mathbf{A}_{(m+2),\,2}\\ &= \mathbf{A}_{1,\,2} \odot \cdots \odot \mathbf{A}_{(m+1),\,2} \odot \mathbf{A}_{(m+2),\,2}. \quad (40)\end{aligned}$$

The recursion finally terminates at $\hat{x}^{(N-1)}$, which we are going to denote by $\overline{x}$

$$\begin{aligned}&\overline{x}_{i_{1,\,3},\, k^{(N-1)},\, l^{(N-1)}}\\ &:= \hat{x}^{(N-1)}_{i_{1,\,3},\, k^{(N-1)},\, l^{(N-1)}}\\ &= \sum_{f=1}^{F} c_f \left(a^{i_{1,\,3}-1}_{f,\,1} d^{(N-1)}_{k^{(N-1)},\, f} e^{(N-1)}_{l^{(N-1)},\, f}\right) \quad (41)\end{aligned}$$

for $k^{(N-1)} = 1, \ldots, \prod_{n=1}^{N} I_{n,1}$, $l^{(N-1)} = 1, \ldots, \prod_{n=1}^{N} I_{n,\,2}$. We have

$$\mathbf{D}^{(N-1)} = \left(d^{(N-1)}_{k^{(N-1)},\, f}\right) = \mathbf{A}_{1,\,1} \odot \cdots \odot \mathbf{A}_{N,\,1}$$
$$\mathbf{E}^{(N-1)} = \left(e^{(N-1)}_{l^{(N-1)},\, f}\right) = \mathbf{A}_{1,\,2} \odot \cdots \odot \mathbf{A}_{N,\,2}. \quad (42)$$

Since $\mathbf{A}_{1,\,3}$ is Vandermonde, Theorem 2 can be invoked to claim uniqueness, provided that

$$I_{1,\,3} + k_{\mathbf{D}^{(N-1)}} + k_{\mathbf{E}^{(N-1)}} \geq 2F + 3 - 1. \quad (43)$$

Similar to the 2-D case, each product form

$$c_f \left(a^{i_{1,\,3}-1}_{f,\,1} d^{(N-1)}_{k^{(N-1)},\, f} e^{(N-1)}_{l^{(N-1)},\, f}\right)$$

can be put in one-to-one correspondence with $c_f \prod_{n=1}^{N} a^{i_n - 1}_{f,\,n}$, $f = 1, \ldots, F$. Therefore, uniqueness of the F rank-one 3-D factors $c_f (a^{i_{1,\,3}-1}_{f,\,1} d^{(N-1)}_{k^{(N-1)},\, f} e^{(N-1)}_{l^{(N-1)},\, f})$ is equivalent to uniqueness of the F rank-one N-D factors $c_f \prod_{n=1}^{N} a^{i_n-1}_{f,\,n}$. It follows that the rank-one factors $c_f \prod_{n=1}^{N} a^{i_n-1}_{f,\,n}$ and, hence, the parameter $(N+1)$-tuples $(a_{f,\,1}, \ldots, a_{f,\,N}, c_f)$, $f = 1, \ldots, F$, are unique, provided that (43) holds true. Invoking Proposition 4, almost-sure uniqueness holds, provided there exist $2N+1$ integers $I_{1,\,3} \geq 2$ and $I_{n,\,j} \geq 2$ for $n = 1, \ldots, N$ and $j = 1, 2$ such that

$$I_{1,3} + \min\left(\prod_{n=1}^{N} I_{n,\,1},\, F\right) + \min\left(\prod_{n=1}^{N} I_{n,\,2},\, F\right) \geq 2F + 2 \quad (44)$$

subject to

$$\begin{aligned}I_{1,\,1} + I_{1,\,2} + I_{1,\,3} &= I_1 + 2\\ I_{n,\,1} + I_{n,\,2} &= I_n + 1, \qquad n = 2, \ldots, N\end{aligned}$$

or, equivalently

$$I_1 - I_{1,\,1} - I_{1,\,2} + \min\left(\prod_{n=1}^{N} I_{n,\,1},\, F\right) + \min\left(\prod_{n=1}^{N} I_{n,\,2},\, F\right) \geq 2F$$

subject to

$$\begin{cases} I_{1,\,1} + I_{1,\,2} \leq I_1 & \\ I_{n,\,1} + I_{n,\,2} = I_n + 1, & n = 2, \ldots, N\\ I_{n,\,j} \geq 2, & \forall\, n,\, j\end{cases}$$

and the proof is complete. □

Proof of Theorem 5: If I_1 is even, pick $I_{1,\,1} = I_{1,\,2} = I_1/2$; otherwise, pick $I_{1,\,1} = (I_1 - 1)/2$ and $I_{1,\,2} = (I_1 + 1)/2$ [thereby satisfying (29)].

If I_n is even, pick $I_{n,\,1} = I_n/2$, $I_{n,\,2} = (I_n + 2)/2$; otherwise, let $I_{n,\,1} = I_{n,\,2} = (I_n + 1)/2$ [hence satisfying (29)] for all $n = 2, \ldots, N$.

Once we pick all $2N$ integers following the above rules, (25) assures that inequality (28) holds. Invoking Proposition 5 completes the proof. □

Proof of Proposition 2: It suffices to show that when $IJ = F$

$$\begin{aligned}&H(\alpha_1, \alpha_2, \ldots, \alpha_F, \beta_1, \beta_2, \ldots, \beta_{F-1})\\ &\quad = \det(\mathbf{A}(\alpha_1, \alpha_2, \ldots, \alpha_F) \odot \mathbf{B}(\beta_1, \beta_1, \ldots, \beta_{F-1}))\end{aligned}$$

is a nontrivial analytic function in $\mathbb{C}^{2F-1}$, where both $\mathbf{A}$ and $\mathbf{B}$ are Vandermonde matrices defined by (2). For any given I, J, F with $2 \leq I \leq F$, $2 \leq J \leq F$, and $IJ = F$, define the generators $\alpha_1 = 0$, $\alpha_f = e^{\sqrt{-1}(2\pi/(F-1))J(f-2)}$ for $f = 2, \ldots, F$ and $\beta_f = e^{\sqrt{-1}(2\pi/(F-1))(f-1)}$ for $f = 1, \ldots, F-1$. It can be verified that with this choice of generators for $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & 1 & e^{\sqrt{-1}(2\pi/(F-1))} & \cdots & e^{\sqrt{-1}(2\pi/(F-1))(F-2)}\\ 1 & 1 & e^{\sqrt{-1}(2\pi/(F-1))2} & \cdots & e^{\sqrt{-1}(2\pi/(F-1))(F-3)}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & e^{\sqrt{-1}(2\pi/(F-1))(J-1)} & \cdots & e^{\sqrt{-1}(2\pi/(F-1))(F-J)}\\ 0 & 1 & e^{\sqrt{-1}(2\pi/(F-1))J} & \cdots & e^{\sqrt{-1}(2\pi/(F-1)(F-J-1)}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 1 & e^{\sqrt{-1}(2\pi/(F-1))(F-2)} & \cdots & e^{\sqrt{-1}(2\pi/(F-1))}\\ 0 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

which is full rank; hence, H is nontrivial in $\mathbb{C}^{2F-1}$. □

Proof of Proposition 3: Assume $I \leq J$, without loss of generality. Spawn two dimensions out of J: $J_1 = J-1$, $J_2 = 2$. Collapse I and $J-1$. We are now in $2 \times I(J-1) \times K$ 3-D space, with the dimension corresponding to $I(J-1)$ being full k-rank almost surely. Theorem 2 then yields $2F + 2 \leq \min(IJ - I, F) + k_{\mathbf{S}} + 2$. Since $k_{\mathbf{S}} = F$ has been assumed, the desired result follows. □

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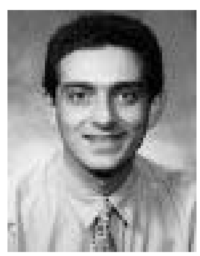
Tao Jiang (S'01) received the B.S. degree from Peking University, Beijing, China, in 1997 and the M.S. degree from the University of Minnesota, Minneapolis, in 2000, both in mathematics. He is currently pursuing the Ph.D. degree with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis.

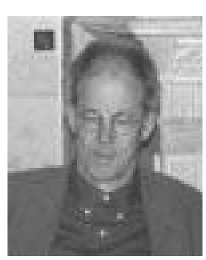
His research interests are in the area of signal processing for communications with focus on wireless communications.



Nicholas D. Sidiropoulos (M'92–SM'99) received the Diploma in electrical engineering from the Aristotelian University of Thessaloniki, Thessaloniki, Greece, and the M.S. and Ph.D. degrees in electrical engineering from the University of Maryland, College Park (UMCP), in 1988, 1990, and 1992, respectively.

From 1988 to 1992, he was a Fulbright Fellow and a Research Assistant at the Institute for Systems Research (ISR), University of Maryland. From September 1992 to June 1994, he served his military service as a Lecturer in the Hellenic Air Force Academy. From October 1993 to June 1994, he also was a member of the technical staff, Systems Integration Division, G-Systems Ltd., Athens, Greece. He held Postdoctoral (from 1994 to 1995) and Research Scientist (from 1996 to 1997) positions at ISR-UMCP before joining the Department of Electrical Engineering, University of Virginia, Charlottesville, in July 1997 as an Assistant Professor. He is currently an Associate Professor with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis. His current research interests are primarily in multi-way analysis and its applications in signal processing for communications and networking.

Dr. Sidiropoulos is a member of the Signal Processing for Communications Technical Committee (SPCOM-TC) of the IEEE Signal Processing Society and currently serves as Associate Editor for IEEE TRANSACTIONS ON SIGNAL PROCESSING and IEEE SIGNAL PROCESSING LETTERS. He received the NSF/CAREER award (Signal Processing Systems Program) in June 1998.



Jos M. F. ten Berge received the masters degree in psychology in 1969 and the Ph.D. degreee in psychometrics in 1977 from the University of Groningen, Groningen, The Netherlands.

He obtained a full professorship in psychometrics in 1988 from the University of Groningen. He is a board member of the Research School IOPS and an Associate Editor of *Psychometrika*. His research areas are psychometrics and three-way analysis.