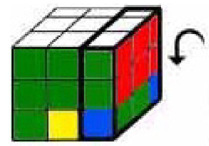

Links Between Multidimensional Low Rank and Harmonic Decomposition

*N. Sidiropoulos**

* *ECE Department, U. of Minnesota, Minneapolis, MN*
nikos@ece.umn.edu

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Roadmap

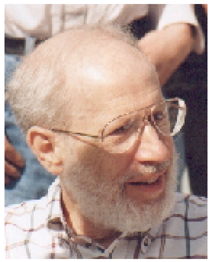
Constantin Caratheodory

1911 → Uniqueness of Harmonic Parameterization in 1-D

2000 ↑

Uniqueness of low-rank Decomposition in N-D → 2000 → Harmonics In N-D

1999 ↑



Joseph Kruskal

1977 → Uniqueness of low-rank decomposition in 3-D

Outline

- ☞ Motivation
- ☞ Carathéodory / Pisarenko
- ☞ Low-rank decomposition of N-way arrays (aka tensors)
- ☞ Kruskal
- ☞ Main results, #1: Generalizing Kruskal to N-D
- ☞ Harmonic prelude: Carathéodory from N-D Kruskal
- ☞ Main results, #2: Generalizing Carathéodory to N-D
- ☞ Conclusions

Harmonics everywhere

☞ 1-D:

- Periodicities in time series: econometrics, astronomy;
- Communications: Synchronization; Radar;
- Seismic; Speech; Medicine (EEG, ECG); ...

☞ N-D:

- Joint AZ-EL-Delay-Doppler in SAP for COM and Radar;
- Multicarrier COM; fluorescence excitation-emission matrices;
- Displacement estimation in N dimensions; image modeling; ...

Spectral analysis

☞ Line Spectra Analysis Problem: Given a finite number of measurements

$$x_i = \sum_{f=1}^{F} c_f e^{j\omega_f (i-1)} + n_i, \quad i = 1, \cdots, I,$$

find $\omega_f \in [-\pi, \pi)$, $f = 1, \cdots, F$

☞ FFT, Windows, Filterbanks, Pisarenko, Capon, MUSIC, ESPRIT, ...

☞ Underpinning technique: uniqueness of harmonic parameterization

Carathéodory / Pisarenko

☞ Any psd $I \times I$ Toeplitz $\mathbf{T}$ of rank $F < I$ can be uniquely decomposed as

$$\mathbf{T} = \mathbf{V}\mathbf{D}\mathbf{V}^H,$$

$\mathbf{V}:\ I \times F$

$\mathbf{v}_f = [1\ e^{j\omega_f} \cdots e^{j\omega_f(I-1)}]^T,\ f = 1, \cdots, F$

$\{\omega_f \in [-\pi, \pi)\}_{f=1}^{F}$ distinct

$\mathbf{D}$ (diag) contains positive reals.

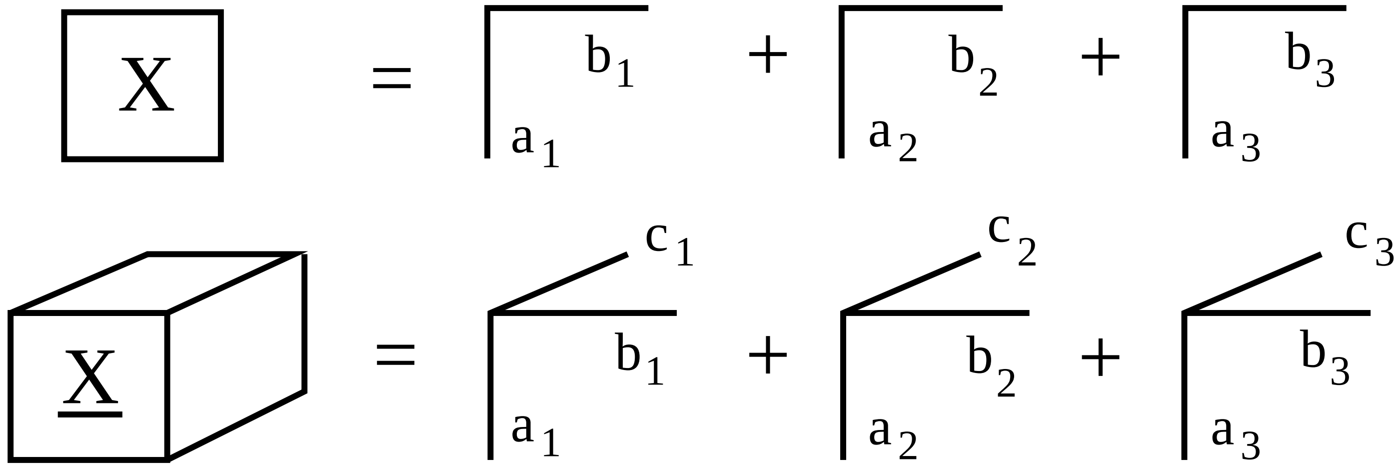
Carathéodory / Pisarenko, Cont.

☞ Theorem: (*Carathéodory's Uniqueness Result*, c. 1911) Given

$$x_i = \sum_{f=1}^{F} |c_f|^2 e^{j\omega_f (i-1)}, \quad i = 1, \cdots, I,$$

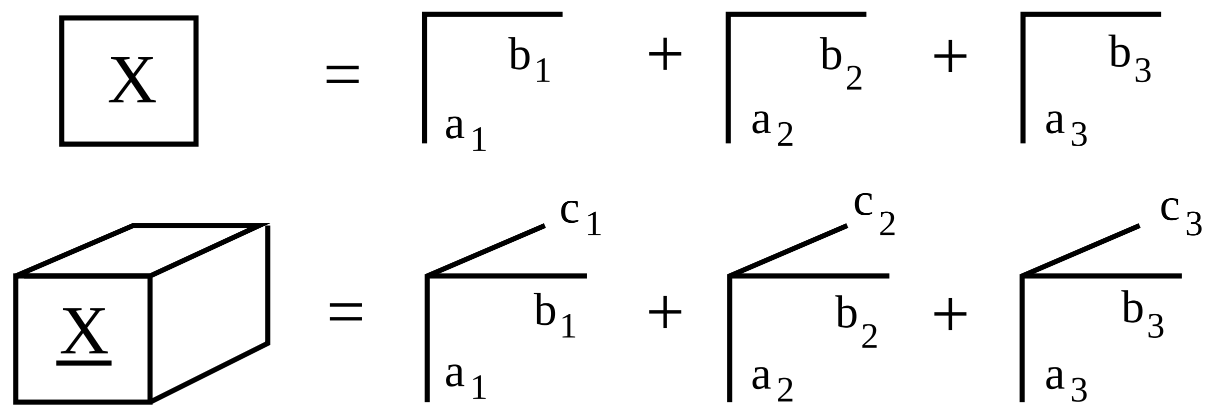
if $I \geq F+1$, then $\omega_f \in [-\pi, \pi)$ and $|c_f|^2$, $f = 1, \cdots, F$ are unique.

PARAFAC/CANDECOMP



- Fact 1: Low-rank matrix (2-way array) decomposition not unique
- Fact 2: Low-rank 3- and higher-way array decomposition (PARAFAC) is unique :-)

Backbone



☞ Kruskal 1977, $N = 3, \mathbb{R}$: $k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} \geq 2F + 2$

☞ Sidiropoulos & Bro J. Chemo 2000: any N, $\mathbb{C}$:
$\sum_{n=1}^{N} k - ranks \geq 2F + (N-1)$

☞ k-rank: $k_{\mathbf{A}}$:= maximum r such that *every* r columns of $\mathbf{A}$ are linearly independent

Proof of Theorem (sketch)

☞ Given *quadrilinear* model of rank F

$$x_{i,j,l,m} = \sum_{f=1}^{F} a_{i,f} b_{j,f} g_{l,f} h_{m,f}$$

☞ Unfolding into 3-D (*matricizing* [Kiers])

$$y_{i,j,k} := x_{i,j,\lceil \frac{k}{M} \rceil, k-\left(\lceil \frac{k}{M} \rceil - 1\right)M}$$

$$i = 1,\cdots,I,\ j = 1,\cdots,J,\ k = 1,\cdots,ML$$

☞ Define $ML \times F$ matrix $\mathbf{C}$

$$c_{k,f} := g_{\lceil \frac{k}{M} \rceil, f} h_{k-\left(\lceil \frac{k}{M} \rceil - 1\right)M, f} \quad key: \mathbf{C} = \mathbf{G} \odot \mathbf{H}$$

Instrumental Lemma

Lemma: (*k-rank of KRP - Sidiropoulos & Liu, 1999*) $\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_F] \in \mathbb{C}^{I \times F}$, $\mathbf{B} = [\mathbf{b}_1, \cdots, \mathbf{b}_F] \in \mathbb{C}^{J \times F}$. If $k_{\mathbf{A}} \geq 1$ and $k_{\mathbf{B}} \geq 1$, then it holds that

$$k_{\mathbf{B} \odot \mathbf{A}} \geq \min(k_{\mathbf{A}} + k_{\mathbf{B}} - 1, F),$$

whereas if $k_{\mathbf{A}} = 0$ and/or $k_{\mathbf{B}} = 0$

$$k_{\mathbf{B} \odot \mathbf{A}} = 0$$

Proof of Theorem (sketch)

- Apply 3-way result & Lemma:

$$k_{\mathbf{A}} + k_{\mathbf{B}} + \min\left(k_{\mathbf{G}} + k_{\mathbf{H}} - 1, F\right) \geq 2F + 2$$

- WLOG WMA $k_{\mathbf{A}} \geq k_{\mathbf{B}} \geq k_{\mathbf{G}} \geq k_{\mathbf{H}}$
- If $k_{\mathbf{G}} + k_{\mathbf{H}} > F + 1$ then $k_{\mathbf{A}} + k_{\mathbf{B}} \geq k_{\mathbf{G}} + k_{\mathbf{H}} > F + 1 \Rightarrow$ $k_{\mathbf{A}} + k_{\mathbf{B}} \geq F + 2$, and hence

$$k_{\mathbf{A}} + k_{\mathbf{B}} + \min\left(k_{\mathbf{G}} + k_{\mathbf{H}} - 1, F\right) \geq F + 2 + F = 2F + 2$$

- else if $k_{\mathbf{G}} + k_{\mathbf{H}} \leq F + 1$ then $\rightarrow$

$$k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{G}} + k_{\mathbf{H}} \geq 2F + 3$$

- Induction for $N > 4$

Discussion

- Stronger than Kruskal applied to 3-D slices: matching issue
- Even disregarding the matching issue, consider:
- $k_{\mathbf{A}} = k_{\mathbf{B}} = k_{\mathbf{G}} = 2$, $k_{\mathbf{H}} = 3$, and $F = 3$
- Kruskal: $2+2+3 = 7 < 2\times 3+2 = 8$:-(
- New result: $2+2+2+3 = 9 = 2\times 3+3$:-)
- Full k-rank:

$$\sum_{n=1}^{N} \min(I_n, F) \geq 2F + (N-1)$$

- Higher N, better ID for given rank F
- In $N \geq 2F-1$ dimensions, decomposition of rank-F arrays a.s. unique

Harmonic Prelude: 1-D harmonics (including damping and phase)

☞ Theorem: Given

$$x_i = \sum_{f=1}^{F} c_f a_f^{i-1}, \quad i = 1, \cdots, I,$$

with $c_f \in \mathbb{C}$ and $a_f \in \mathbb{C}$, if $I \geq 2F$ then (a_f, c_f), $f = 1, \cdots, F$ unique.

Proof

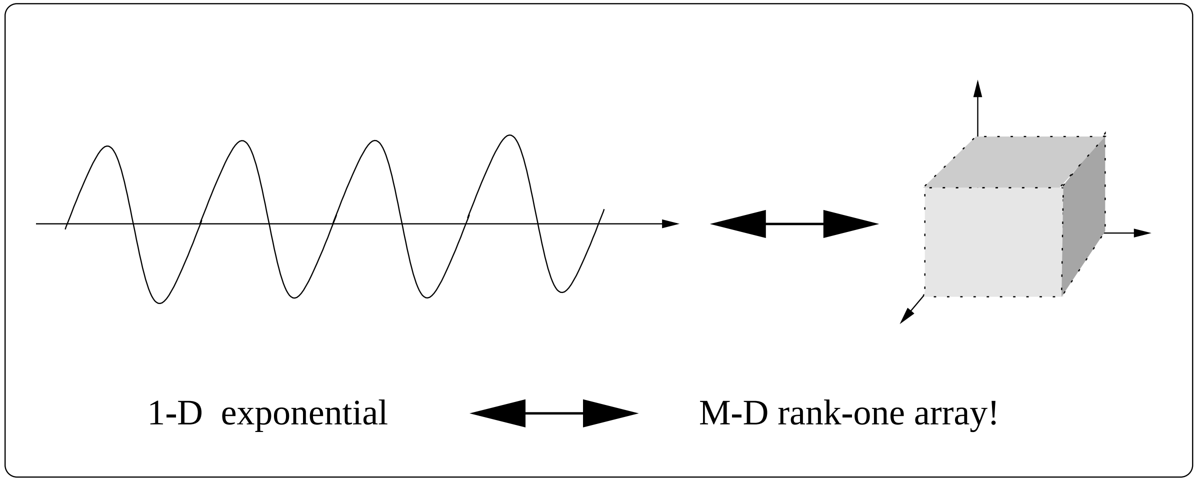
Define the M-way array

$$\bar{x}_{i_1,\cdots,i_M} := x_{i_1+\cdots+i_M-(M-1)} = \sum_{f=1}^{F} c_f a_f^{i_1+\cdots+i_M-M} = \sum_{f=1}^{F} c_f a_f^{i_1-1} \cdots a_f^{i_M-1},$$

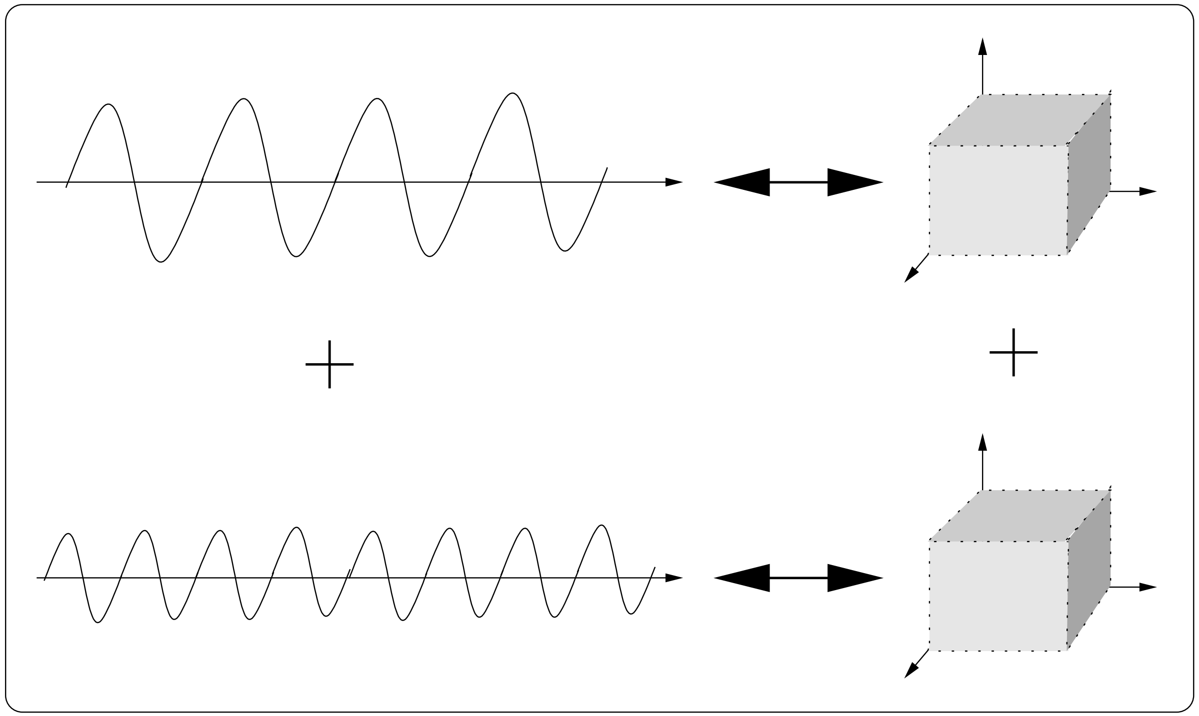
for

$$\left.\begin{array}{c} i_1 = 1,\cdots,I_1 \\ \vdots \\ i_M = 1,\cdots,I_M \end{array}\right\}, \quad I_1+\cdots+I_M = I+M-1.$$

Multidimensional embedding: $a^{x+y} = a^x a^y \Longrightarrow$



Multidimensional embedding, Cont.



Proof, Cont.

☞ Unique, provided

$$\sum_{m=1}^{M} \min(I_m, F) \geq 2F + (M-1).$$

☞ Pick $M = I - 1$ and $I_m = 2$ for all m (this choice actually maximizes $\sum_{m=1}^{M} \min(I_m, F)$ for all $F > 1$). Then the identifiability condition becomes

$$2(I-1) \geq 2F + (I-2),$$

or, equivalently,

$$I \geq 2F,$$

and thus the proof is complete

Recovering Carathéodory: Conjugation & folding

☞ Given

$$x_i = \sum_{f=1}^{F} |c_f|^2 e^{j\omega_f (i-1)}, \quad i = 1, \cdots, I,$$

$$y_i := x_{I-i+1}^* = \sum_{f=1}^{F} c_f e^{j\omega_f (i-1)}, \quad i = 1, \cdots, I, \; c_f := |c_f|^2 e^{j\omega_f (-I+1)};$$

$$z_i := x_{i+1} = \sum_{f=1}^{F} c_f e^{j\omega_f (I+i-1)}, \quad i = 1, \cdots, I-1,$$

So we get samples of the original model from $i = 1$ $(i-1=0)$ to $i = 2I-1$ $(i-1 = 2I-2)$; from the previous result, unique if $2I - 1 \geq 2F \iff I \geq F + \frac{1}{2} \iff I \geq F+1$ (since integer) - Carathéodory

N-D

☞ For $N \geq 3$ uniqueness by means of N-way Theorem, however, better result possible (also for $N = 2$)

Theorem: (*Sidiropoulos, IEEE Trans. IT, 2001*) Given a sum of F exponentials in N-dimensions

$$x_{i_1,\cdots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n - 1},$$

for $i_n = 1,\cdots,I_n \geq 2$, $n = 1,\cdots,N$, with $c_f \in \mathbb{C}$ and $a_{f,n} \in \mathbb{C}$ such that $a_{f_1,n} \neq a_{f_2,n}$, $\forall f_1 \neq f_2$ and all n, if

$$\sum_{n=1}^{N} I_n \geq 2F + (N-1),$$

then there exist unique $(a_{f,n}, c_f)$, $n = 1,\cdots,N$ that give rise to $x_{i_1,\cdots,i_N}$.

Proof

Define the extended multi-way array

$$\bar{x}_{i_{1,1},\cdots,i_{1,I_1-1},\cdots,i_{N,1},\cdots,i_{N,I_N-1}} := x_{i_{1,1}+\cdots+i_{1,I_1-1}-(I_1-2),\cdots,i_{N,1}+\cdots+i_{N,I_N-1}-(I_N-2)} =$$

$$\sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_{n,1}+\cdots+i_{n,I_n-1}-(I_n-1)} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_{n,1}-1} \cdots a_{f,n}^{i_{n,I_n-1}-1} =$$

$$\sum_{f=1}^{F} c_f \prod_{n=1}^{N} \prod_{m=1}^{I_n-1} a_{f,n}^{i_{n,m}-1}, \quad i_{n,m} \in \{1,2\}, \forall n,m.$$

☞ From N-way Theorem, and the fact that Vandermonde matrices have full k-rank, $\rightarrow$ the rank-one factors $c_f \prod_{n=1}^{N} \prod_{m=1}^{I_n-1} a_{f,n}^{i_{n,m}-1}$ and hence $a_{f,n}$, $n = 1,\cdots,N$ and c_f, $f = 1,\cdots,F$ are unique provided that $\rightarrow$

Proof, Cont.

$$\sum_{n=1}^{N} \sum_{m=1}^{I_n - 1} 2 \geq 2F + \left(\sum_{n=1}^{N} (I_n - 1) \right) - 1$$

- ☞ Note that the sum on the right hand side is the total number of effective dimensions
- ☞ Equivalently, uniqueness holds provided

$$\sum_{n=1}^{N} I_n \geq 2F + (N - 1)$$

which completes the proof

P-a.s. Uniqueness: Key Result

☞ Theorem: (*Jiang, ten Berge, Sidiropoulos, IEEE Trans. SP, 2001*) For a pair of Vandermonde matrices $\mathbf{A} \in \mathbb{C}^{I \times F}$ and $\mathbf{B} \in \mathbb{C}^{J \times F}$

$$r_{\mathbf{A} \odot \mathbf{B}} = k_{\mathbf{A} \odot \mathbf{B}} = min(IJ, F), \quad P_{\mathcal{L}}(\mathbb{C}^{2F}) - a.s.$$

☞ Corollary: For a pair of matrices $\mathbf{A} \in \mathbb{C}^{I \times F}$ and $\mathbf{B} \in \mathbb{C}^{\mathbf{J} \times \mathbf{F}}$,

$$r_{\mathbf{A} \odot \mathbf{B}} = k_{\mathbf{A} \odot \mathbf{B}} = min(IJ, F), \quad P_{\mathcal{L}}(\mathbb{C}^{(I+J)F}) - a.s.$$

P-a.s. Uniqueness: Damped Exponentials

☞ Theorem (*Jiang, ten Berge, Sidiropoulos, IEEE Trans. SP, 2001*) Given

$$x_{k,l} = \sum_{f=1}^{F} c_f a_f^{k-1} b_f^{l-1}, \quad k = 1, \ldots, K \geq 4, \quad l = 1, \ldots, L \geq 4,$$

if

$$F \leq \left\lfloor \frac{K}{2} \right\rfloor \left\lceil \frac{L}{2} \right\rceil,$$

and $P_{\mathcal{L}}(\mathbb{C}^{2F})$ (the distribution used to draw the $2F$ complex exponential parameters (a_f, b_f), $f = 1, \ldots, F$) is continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$, then the parameter triples (a_f, b_f, c_f), $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{2F})$ almost surely unique.

P-a.s. Uniqueness: Undamped Exponentials

☞ Theorem (*Liu, Sidiropoulos, IEEE Trans. SP, 2002*) Given

$$x_{k,l} = \sum_{f=1}^{F} c_f e^{j\omega_f (k-1)} e^{j\nu_f (l-1)}, \quad k = 1, \ldots, K \geq 3, \quad l = 1, \ldots, L \geq 3,$$

if

$$F \leq \left\lceil \frac{K}{2} \right\rceil \left\lceil \frac{L}{2} \right\rceil,$$

and $P_{\mathcal{L}}(\Pi^{2F})$ (the distribution used to draw the $2F$ frequencies (ω_f, ν_f), $f = 1, \ldots, F$) is continuous with respect to the Lebesgue measure in Π^{2F}, then the parameter triples (ω_f, ν_f, c_f), $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\Pi^{2F})$-a.s. unique.

☞ Trick: 2-D conjugate folding

P-a.s. Uniqueness

- Can be extended to N-D harmonic decomposition, see *Sidiropoulos et al, IEEE Trans. SP 2001-2002*
- Conditions close to $\frac{eqns}{unknowns}$ in 2-D

Concluding remarks

- $\exists$ retrieval algorithms (EVD) that work under ID only
- $\exists$ "ML" iterative algorithms that perform close to CRB even at moderate sample sizes and SNRs
- Necessity of best known ID conditions?
- Fast algorithms that perform close to ML?