D. Proof of Theorem 3

From regularity condition (3.1), the asymptotic behaviors of $\boldsymbol{f}_n$ and $\boldsymbol{R}_n$ are directly related. The standard result on regular functions of asymptotically normal statistics (see, e.g., [12, Theorem, p. 22]) applies. So (3.3) with $\boldsymbol{C}_f = \boldsymbol{D}^{\text{alg}}_{f,R}\boldsymbol{C}_R(\boldsymbol{D}^{\text{alg}}_{f,R})^H$. Furthermore, this closed-form expression simplifies if (1) and (2) are taken into account

$$\begin{aligned}\boldsymbol{f} &= \text{alg}(\boldsymbol{E}(\boldsymbol{f})(\boldsymbol{\Delta}+\delta\boldsymbol{\Delta})\boldsymbol{E}^H(\boldsymbol{f})+(c_u+\delta c_u)\boldsymbol{B}\boldsymbol{B}^H)\\ &= \boldsymbol{f}+\boldsymbol{D}^{\text{alg}}_{f,R}\text{Vec}(\boldsymbol{E}(\boldsymbol{f})\delta\boldsymbol{\Delta}\boldsymbol{E}^H(\boldsymbol{f})+\delta c_u\boldsymbol{B}\boldsymbol{B}^H)+o(\delta\boldsymbol{\Delta})+o(\delta c_u)\\ &= \boldsymbol{f}+\boldsymbol{D}^{\text{alg}}_{f,R}\left(\sum_{k=1}^{K}\delta a_k^2(\boldsymbol{e}(f_k)\otimes_c \boldsymbol{e}^H(f_k))+\delta c_u\text{Vec}(\boldsymbol{B}\boldsymbol{B}^H)\right)\\ &\quad +o(\delta\boldsymbol{\Delta})+o(\delta c_u) \end{aligned} \tag{A9}$$

where

$$\text{Vec}(\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)) = \boldsymbol{e}(f_k)\otimes_c \boldsymbol{e}^H(f_k)$$

is used in the third equality. Therefore, the following constraints upon $\boldsymbol{D}^{\text{alg}}_{f,R}$ hold:

$$\boldsymbol{D}^{\text{alg}}_{f,R}[\boldsymbol{e}(f_k)\otimes_c \boldsymbol{e}^H(f_k)] = \boldsymbol{0}, \qquad k=1,\ldots,K \quad \text{and} \quad \boldsymbol{D}^{\text{alg}}_{f,R}\text{Vec}(\boldsymbol{B}\boldsymbol{B}^H) = \boldsymbol{0} \tag{A10}$$

and using (2.8), the proof follows. □

References

[1] E. J. Hannan, *Multiple Time Series*. New York: Wiley, 1970.
[2] P. J. Brockwell and R. A. Davis, *Time Series, Theory and Methods*, 2nd ed. New York: Springer-Verlag, 1991.
[3] B. Porat, *Digital Processing of Random Signals, Theory and Methods*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
[4] M. Rosenblatt, *Stationary Sequences and Random Fields*. Basel, Switzerland: Birkhäuser, 1985.
[5] T. H. Li, B. Kedem, and S. Yakowitz, "Asymptotic normality of sample autocovariances with an application in frequency estimation," *Stochastic Processes Their Applic.*, vol. 52, pp. 329–349, 1994.
[6] B. Picinbono, "Second-order complex random vectors and normal distributions," *IEEE Trans. Signal Processing*, vol. 44, pp. 2637–2640, Oct. 1996.
[7] F. D. Neeser and J. L. Massey, "Proper complex random processes with applications to information theory," *IEEE Trans. Inform. Theory*, vol. 39, pp. 1293–1302, July 1993.
[8] P. Billingsley, *Probability and Measure*, 2nd ed. New York: Wiley, 1986.
[9] A. Gorokhov, Y. Abramovich, and J. F. Böhme, "Unified analysis of DOA estimation algorithms for covariance matrix transforms," *Signal Processing*, vol. 55, pp. 107–115, 1996.
[10] P. Stoica, T. Söderströom, and F. Ti, "Overdetermined Yule–Walker estimation of the frequencies of multiple sinusoids: Accuracy aspects," *Signal Processing*, vol. 16, pp. 155–174, 1989.
[11] J. F. Cardoso and E. Moulines, "Asymptotic performance analysis of direction-finding algorithms based on fourth-order cumulants," *IEEE Trans. Signal Processing*, vol. 43, pp. 214–224, Jan. 1995.
[12] R. J. Serfling, *Approximation Theorems of Mathematical Statistics*. New York: Wiley, 1980.
[13] P. Stoica and T. Söderström, "Statistical analysis of MUSIC and subspace rotation estimates of sinusoidal frequencies," *IEEE Trans. Signal Processing*, vol. 39, pp. 1836–1847, Aug. 1991.
[14] G. Bienvenu, "Influence of the spatial coherence of the background noise on high resolution passive methods," in *Proc. ICASSP*, Washington, DC, Apr. 1979, pp. 306–309.
[15] S. Orfanidis, *Optimum Signal Processing, an Introduction*. New York: Macmillan, 1988.

Generalizing Carathéodory's Uniqueness of Harmonic Parameterization to N Dimensions

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***Abstract*—Consider a sum of F exponentials in N dimensions, and let I_n be the number of equispaced samples taken along the nth dimension. It is shown that if the frequencies or decays along every dimension are distinct and $\sum_{n=1}^{N} I_n \geq 2F + (N-1)$, then the parameterization in terms of frequencies, decays, amplitudes, and phases is unique. The result can be viewed as generalizing a classic result of Carathéodory to N dimensions. The proof relies on a recent result regarding the uniqueness of low-rank decomposition of N-way arrays.**

***Index Terms*—Multidimensional harmonic retrieval, multiway analysis, PARAllel FACtor (PARAFAC) analysis, spectral analysis, uniqueness.**

I. Introduction

The problem of harmonic retrieval and, more generally, exponential retrieval permeates the applied sciences and engineering. Although one-dimensional (1-D) exponential retrieval is most common (e.g., see [17] and references therein), the multidimensional case appears in a variety of important applications like joint azimuth, elevation, delay, and Doppler estimation in antenna array processing for communications [3]–[6], synthetic aperture radar (e.g., [7], [10] and references therein), and also certain signal separation problems in chemistry.

A wide variety of nonparametric and parametric techniques have been developed for the harmonic retrieval problem in one or more dimensions. Underpinning technique and practice of harmonic retrieval is the issue of identifiability, i.e., uniqueness of model parameterization. Owing to the work of Carathéodory [1] and later Pisarenko [11], this issue is well understood for the case of 1-D harmonics. In the case of multidimensional harmonics (and, more generally, exponentials), one can apply the 1-D result separately in each dimension, but this has two serious drawbacks. First, this approach does not reap the benefits of the rich multidimensional structure, leading to uniqueness conditions that are unnecessarily strict. Second, the association problem (i.e., whether the "pairing" of frequencies along different dimensions is unique) remains.

The uniqueness problem is hard for harmonics in two or higher dimensions. Only partial results are known for the two–dimensional (2-D) case [8], [10]. For example, [10] considers one possible formulation of the 2-D harmonic retrieval problem wherein the frequencies are assumed to occur at the intersections of certain unknown grid lines in the 2-D frequency domain, and provides sufficient conditions for identifiability. In the case of a single realization of the 2-D harmonic mixture, the conditions in [10] require that one has sufficiently many samples in each dimension for the 1-D result of Carathéodory to kick in.

The contribution of this correspondence is a general uniqueness result for N-dimensional exponential mixtures that is valid for any N and

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improves with increasing N. The adopted formulation is tailored toward arbitrary[1] distribution of the unknown N-dimensional exponential parameters, as in [3]. The proof relies on—and actually improves for the special case of mixtures of N-dimensional exponentials—a recent result [12], [13] regarding uniqueness of low-rank decomposition of N-way arrays. Low-rank decomposition of N-way arrays is known under the common name PARAllel FACtor analysis, or PARAFAC for short.

A. Organization

The rest of this correspondence is structured as follows. Following a brief discussion of notation and other preliminaries, Section II summarizes the basic theorem of [12], [13] on the uniqueness of low-rank decomposition of N-way arrays. Section III contains a brief statement of the Carathéodory parameterization result. Section IV is a 1-D preamble to Section V, which contains the main result on the uniqueness of parameterization of exponential mixtures in N dimensions. Conclusions are drawn in Section VI.

B. Notation and Some Preliminaries

$\mathbb{R}$ stands for the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. Matrices (vectors) are denoted by boldface capital (lower case) letters. ^{T}, ^{H} stand for transpose and Hermitian transpose, respectively. The symbol j is reserved for $\sqrt{-1}$. N denotes the number of dimensions, whereas I_n denotes the number of (equispaced) samples along the nth dimension. An N-dimensional (also known as N-way) array is a data set that is indexed by N indexes: $x_{i_1,\ldots,i_N}$, where

$$i_n \in \{1, \ldots, I_n\}, \qquad n = 1, \ldots, N.$$

The rank of a matrix is the smallest number of rank-one matrices needed to decompose the given matrix into a sum of rank-one factors. Each rank-one factor is the outer product of two vectors. Similarly, the rank of an N-way array is defined as the smallest number of rank-one N-way factors needed to decompose it [9]. Each rank-one N-way factor is the "outer product" of N vectors, meaning that its $(i_1, \ldots, i_N)$th element is given by $a_{f,1,i_1} \cdots a_{f,N,i_N}$, where f is a factor index. Thus, an N-way array of rank F can be written as

$$x_{i_1,\ldots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n,i_n}.$$

A constant-envelope 1-D discrete-time exponential is written as $x_i = ce^{j\omega(i-1)}$, $i = 1, \ldots, I$, where $c \in \mathbb{C}$ absorbs both amplitude and phase. A nonconstant-envelope 1-D exponential is written as $x_i = ca^{i-1}$, $i = 1, \ldots, I$, where $a \in \mathbb{C}$ absorbs both decay (or growth) rate and frequency. A 2-D exponential is simply the product of two 1-D exponentials indexed by different independent variables, i.e.,

$$x_{i_1,i_2} = ca_1^{i_1-1}a_2^{i_2-1}. \qquad i_1 = 1, \ldots, I_1, \;\; i_2 = 1, \ldots, I_2$$

and so on and so forth in higher dimensions.

II. PARAFAC

Definition 1: The k-rank of a matrix $\boldsymbol{A}$ (denoted by $k_{\boldsymbol{A}}$) is r iff *every* r columns of $\boldsymbol{A}$ are linearly independent, and either $\boldsymbol{A}$ has r columns or $\boldsymbol{A}$ contains a set of $r+1$ linearly dependent columns. Note that k-rank is always less than or equal to rank.

[1]See [10] for a discussion of "grid" versus "nongrid" 2-D harmonic retrieval problem formulations.

Remark 1: An $m \times \rho$ Vandermonde matrix

$$\boldsymbol{V} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_\rho \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_\rho^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{m-1} & \alpha_2^{m-1} & \cdots & \alpha_\rho^{m-1} \end{bmatrix}$$

with distinct nonzero generators $\alpha_1, \alpha_2, \ldots, \alpha_\rho \in \mathbb{C}$ is full k-rank: $k_{\boldsymbol{V}} = r_{\boldsymbol{V}} = \min(m, \rho)$ [16].

Theorem 1 (Uniqueness of Low-Rank Decomposition of N-Way Arrays [12], [13]): Consider the F-component N-linear model

$$x_{i_1,\ldots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n,i_n}$$

for $i_n = 1, \ldots, I_n \geq 2$, $n = 1, \ldots, N$, with $c_f \in \mathbb{C}$, $a_{f,n,i_n} \in \mathbb{C}$, and suppose that it is irreducible ($\Leftrightarrow$ the rank of the N-way array with typical element $x_{i_1,\ldots,i_N}$ is F). Let $\boldsymbol{A}^{(n)}$ denote the $I_n \times F$ matrix with (i_n, f) element a_{f,n,i_n}. If

$$\sum_{n=1}^{N} k_{\boldsymbol{A}^{(n)}} \geq 2F + (N-1)$$

then given the N-way array $x_{i_1,\ldots,i_N}$, $i_n = 1, \ldots, I_n$, $n = 1, \ldots, N$, its F rank-one N-way factors

$$c_f \prod_{n=1}^{N} a_{f,n,i_n}, \qquad f = 1, \ldots, F$$

are unique.

A basic precursor result for $N = 3$ and array elements drawn from $\mathbb{R}$ is due to Kruskal [9]. Kruskal's result was subsequently generalized to the complex case in [15] (again for $N = 3$); see also [14] for additional results in the context of sensor array processing.

III. Carathéodory's Uniqueness Result

The Carathéodory parameterization result [1] (see also [11]; [17] is a readily accessible general reference) states that any positive semidefinite $I \times I$ Toeplitz matrix $\boldsymbol{T}$ of rank $F < I$ can be uniquely decomposed as

$$\boldsymbol{T} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^H$$

where $\boldsymbol{V}$ is $I \times F$ with fth column

$$\boldsymbol{v}_f = [1 \; e^{j\omega_f} \cdots e^{j\omega_f(I-1)}]^T, \;\; f = 1, \ldots, F, \;\; \{\omega_f \in [-\pi, \pi)\}_{f=1}^{F}$$

are distinct, and $\boldsymbol{D}$ is a diagonal matrix containing positive reals along its diagonal. This result is the basis behind subspace line spectra estimation: cf. [11], [17], and references therein. An important ramification of the uniqueness part of Carthéodory's parameterization is the following result.

Theorem 2 (Carathéodory's Uniqueness Result): Given

$$x_i = \sum_{f=1}^{F} |c_f|^2 e^{j\omega_f(i-1)}, \qquad i = 1, \ldots, I,$$

if $I \geq F+1$, then $\omega_f \in [-\pi, \pi)$ and $|c_f|^2$, $f = 1, \ldots, F$ are unique.

A proof can be readily derived by constructing a positive semidefinite $I \times I$ Toeplitz matrix whose first column is $[x_1 \cdots x_I]^T$, noting that it admits a decomposition of the form $\boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^H$ for the true frequencies and powers, and invoking the uniqueness part of Carthéodory's parameterization.

Theorem 2 applies to zero-phase, constant-envelope exponentials. As a first step toward extending it to multidimensional exponentials, it is instructive to consider the case of nonzero-phase and nonconstant-envelope 1-D exponentials, and derive a uniqueness result based on Theorem 1.

IV. 1-D Exponentials

Theorem 2: Given

$$x_i = \sum_{f=1}^{F} c_f a_f^{i-1}, \qquad i = 1, \ldots, I,$$

with $c_f \in \mathbb{C}$ and $a_f \in \mathbb{C}$, if $I \geq 2F$ then a_f and c_f, $f = 1, \ldots, F$ are unique.

Proof: We may assume without loss of generality that the c_f's are nonzero and the a_f's are distinct, for otherwise the number of components can obviously be reduced to $F' < F$. Define the M-way array

$$\begin{aligned}\overline{x}_{i_1, \ldots, i_M} &:= x_{i_1 + \cdots + i_M - (M-1)} \\ &= \sum_{f=1}^{F} c_f a_f^{i_1 + \cdots + i_M - M} \\ &= \sum_{f=1}^{F} c_f a_f^{i_1 - 1} \cdots a_f^{i_M - 1},\end{aligned}$$

for

$$\left.\begin{array}{c} i_1 = 1, \ldots, I_1 \\ \vdots \\ i_M = 1, \ldots, I_M \end{array}\right\}, \quad I_1 + \cdots + I_M = I + M - 1.$$

From Theorem 1, and the fact that Vandermonde matrices have full k-rank, it follows that the rank-one factors $c_f a_f^{i_1-1} \cdots a_f^{i_M-1}$ and hence[2] c_f and a_f, $f = 1, \ldots, F$ are unique provided that

$$\sum_{m=1}^{M} \min(I_m, F) \geq 2F + (M-1).$$

Pick $M = I - 1$ and $I_m = 2$ for all m (this choice actually maximizes $\sum_{m=1}^{M} \min(I_m, F)$ for all $F > 1$). Then the identifiability condition becomes

$$2(I-1) \geq 2F + (I-2)$$

or, equivalently,

$$I \geq 2F$$

and thus the proof is complete. □

Remark 2: Note that the (deterministic) identifiability requirement of Theorem 3 meets the equations-versus-unknowns bound (I complex data for $2F$ complex unknowns). A generic (almost sure) identifiability result for the case of nonzero phase, constant-envelope 1-D exponentials can be found in [18].

V. N–Dimensional Exponentials

Next, consider a sum of F exponentials in N dimensions. For $N \geq 3$, uniqueness of parameterization can be claimed by means of Theorem 1, however, a better result[3] is possible which also holds for $N = 2$.

Theorem 4: Given a sum of F exponentials in N dimensions

$$x_{i_1, \ldots, i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n - 1}$$

for $i_n = 1, \ldots, I_n \geq 2$, $n = 1, \ldots, N$, with $c_f \in \mathbb{C}$ and $a_{f,n} \in \mathbb{C}$ such that $a_{f_1, n} \neq a_{f_2, n}$, $\forall f_1 \neq f_2$ and all n, if

$$\sum_{n=1}^{N} I_n \geq 2F + (N-1)$$

then there exist unique $a_{f,n}$, $n = 1, \ldots, N$ and c_f, $f = 1, \ldots, F$ that give rise to $x_{i_1, \ldots, i_N}$.

Proof: Define the extended multiway array

$$\begin{aligned}&\overline{x}_{i_{1,1}, \ldots, i_{1, I_1 - 1}, \ldots, i_{N,1}, \ldots, i_{N, I_N - 1}} \\ &\quad := x_{i_{1,1} + \cdots + i_{1, I_1 - 1} - (I_1 - 2), \ldots, i_{N,1} + \cdots + i_{N, I_N - 1} - (I_N - 2)} \\ &\quad = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_{n,1} + \cdots + i_{n, I_n - 1} - (I_n - 1)} \\ &\quad = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_{n,1} - 1} \cdots a_{f,n}^{i_{n, I_n - 1} - 1} \\ &\quad = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} \prod_{m=1}^{I_n - 1} a_{f,n}^{i_{n,m} - 1}, \qquad i_{n,m} \in \{1, 2\}, \qquad \forall n, m.\end{aligned}$$

From Theorem 1, and the fact that Vandermonde matrices have full k-rank, it follows that the rank-one factors

$$c_f \prod_{n=1}^{N} \prod_{m=1}^{I_n - 1} a_{f,n}^{i_{n,m} - 1}$$

and hence $a_{f,n}$, $n = 1, \ldots, N$ and c_f, $f = 1, \ldots, F$ are unique provided that

$$\sum_{n=1}^{N} \sum_{m=1}^{I_n - 1} 2 \geq 2F + \left(\sum_{n=1}^{N} (I_n - 1) \right) - 1.$$

Note that the sum on the right-hand side is the total number of effective dimensions. Equivalently, uniqueness holds provided

$$\sum_{n=1}^{N} I_n \geq 2F + (N-1).$$ □

Remark 3: It is not difficult to see that if $a_{f_1, n^*} = a_{f_2, n^*}$, then the $k_{\boldsymbol{A}}^{(n^*)} = 1$, and hence the n^*th dimension neither contributes to nor takes away from uniqueness: the model (including the parameters along the n^*th dimension) will still be unique, provided

$$\sum_{n=1, n \neq n^*}^{N} I_n \geq 2F + (N-2).$$

Generalizing, if

$$\forall n \in \{1, \ldots, N - N^*\}: a_{f_1, n} \neq a_{f_2, n}, \qquad \forall f_1 \neq f_2$$

and

$$\forall n \in \{N - N^* + 1, \ldots, N\}: \exists f_2(n) \neq f_1(n): a_{f_1(n), n} = a_{f_2(n), n}$$

then

$$\sum_{n=1}^{N - N^*} I_n \geq 2F + (N - N^* - 1)$$

is sufficient for uniqueness.

Remark 4: If an additional M nonexponential dimensions are available

$$x_{i_1, \ldots, i_N, j_1, \ldots, j_M} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} a_{f,n}^{i_n - 1} \prod_{m=1}^{M} b_{f, m, j_m}$$

for $j_m = 1, \ldots, J_m \geq 2$, $m = 1, \ldots, M$, with $b_{f, m, 1} = 1$, $\forall f, m$ by convention, then it is clear from Theorem 1 and the proof of Theorem 4 that uniqueness (including the associated component vectors along nonexponential dimensions) holds provided that

$$\sum_{n=1}^{N} I_n + \sum_{m=1}^{M} k_{\boldsymbol{B}^{(m)}} \geq 2F + (N + M - 1)$$

[2] a_f can be determined from the corresponding rank-one factor by dividing the second element by the first element along any dimension; c_f is equal to the value of the said rank-one factor at $i_1 = \cdots = i_M = 1$.

[3] In order to see this, suppose $N = 3$, $I_1 = 2$, $I_2 = 2$, $I_3 = 128$, and $F < 128$. Then Theorem 1 indicates that only two exponentials can be resolved; whereas Theorem 4 shows that up to 65 exponentials can be resolved.

TABLE I
SAMPLE SIZE DISTRIBUTION AND MINIMUM TOTAL NUMBER OF SAMPLES FOR $N = 2$

F	$2F + (N-1) = 2F + 1$	I_1	I_2	$S_{total} = I_1 \times I_2$
2	5	2	3	6
3	7	2	5	10
4	9	2	7	14
5	11	2	9	18

TABLE II
SAMPLE SIZE DISTRIBUTION AND MINIMUM TOTAL NUMBER OF SAMPLES FOR $N = 3$

F	$2F + (N-1) = 2F + 2$	I_1	I_2	I_3	$S_{total} = I_1 \times I_2 \times I_3$
2	6	2	2	2	8
3	8	2	2	4	16
4	10	2	2	6	24
5	12	2	2	8	32

where $\boldsymbol{B}^{(m)}$ denotes the $J_m \times F$ matrix with (j_m, f) element b_{f, m, j_m}.

In order to illustrate the power of Theorem 4, Tables I and II provide examples of sample size distributions that guarantee identifiability with smallest total number of samples for various values of F in $N = 2$ and $N = 3$ dimensions.

VI. CONCLUSION

The fact that Theorem 1 can be used to prove Theorem 3 seems quite surprising—the two appear unrelated, and have very different histories. At hindsight, it is clear that what enables this connection (and essentially also Theorem 4) is nothing but the shift property of exponentials, which allows recasting a 1-D exponential in the form of a multidimensional rank-one factor.

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REFERENCES

[1] C. Carathéodory and L. Fejér, "Uber den Zusammenghang der Extemen von harmonischen Funktionen mit ihren Koeffizienten und uber den Picard-Landauschen Satz," *Rendiconti del Circolo Matematico di Palermo*, vol. 32, pp. 218–239, 1911.

[2] T. Georgiou, "Spectral Estimation via Selective Harmonic Amplification," *IEEE Trans. Automat. Contr.*, to be published.

[3] M. Haardt and J. H. Nossek, "Simultaneous Schur decomposition of several nonsymmetric matrices to achieve automatic pairing in multidimensional harmonic retrieval problems," *IEEE Trans. Signal Processing*, vol. 46, pp. 161–169, Jan. 1998.

[4] ——, "3-D unitary ESPRIT for joint 2-D angle and carrier estimation," in *Proc. ICASSP-97*, vol. 1, 1997, pp. 255–258.

[5] M. Haardt, C. Brunner, and J. H. Nossek, "Efficient high-resolution 3-D channel sounding," in *Proc. 48th IEEE Vehicular Technology Conf. (VTC '98)*, Ottawa, ON, Canada, May 1998, pp. 164–168.

[6] ——, "Joint estimation of 2-D arrival angles, propagation delays, and Doppler frequencies to determine realistic directional simulation models for smart antennas," in *Proc. IEEE Digital Signal Processing Workshop*, UT, Aug. 1998.

[7] Y. Hua, "High resolution imaging of continuously moving object using stepped frequency radar," *Signal Processing*, vol. 35, pp. 33–40, Jan. 1994.

[8] H. Yang and Y. Hua, "On rank of block Hankel Matrix for 2-D frequency detection and estimation," *IEEE Trans. Signal Processing*, vol. 44, pp. 1046–1048, Apr. 1996.

[9] J. B. Kruskal, "Three-way arrays: Rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics," *Linear Algebra and Its Applications*, vol. 18, pp. 95–138, 1977.

[10] J. Li, P. Stoica, and D. Zheng, "An efficient algorithm for two-dimensional frequency estimation," *Multidim. Syst. Signal Processing*, vol. 7, no. 2, pp. 151–178, Apr. 1996.

[11] V. F. Pisarenko, "The retrieval of harmonics from a covariance function," *Geophys. J. Roy. Astron. Soc.*, vol. 33, pp. 347–366, 1973.

[12] N. D. Sidiropoulos and R. Bro, "On the uniqueness of multilinear decomposition of N-way arrays," *J. Chemometrics*, vol. 14, no. 3, pp. 229–239, May 2000. Special cross-disciplinary issue on multi-way analysis.

[13] ——, "On communication diversity for blind identifiability and the uniqueness of low-rank decomposition of N-way arrays," in *Proc. ICASSP2000*, Istanbul, Turkey, June 5–9, 2000.

[14] N. D. Sidiropoulos, R. Bro, and G. B. Giannakis, "Parallel factor analysis in sensor array processing," *IEEE Trans. Signal Processing*, vol. 48, pp. 2377–2388, Aug. 2000.

[15] N. D. Sidiropoulos, G. B. Giannakis, and R. Bro, "Blind PARAFAC Receivers for DS-CDMA Systems," *IEEE Trans. Signal Processing*, vol. 48, pp. 810–823, Mar. 2000.

[16] N. D. Sidiropoulos and X. Liu, "Identifiability results for blind beamforming in incoherent multipath with small delay spread," *IEEE Trans. Signal Processing*, vol. 49, pp. 221–236, Jan. 2001.

[17] P. Stoica and R. Moses, *Introduction to spectral analysis*. Upper Saddle River, NJ: Prentice-Hall, 1997.

[18] P. Stoica and T. Söderström, "Parameter identifiability problem in signal processing," *Proc. Inst. Elec. Eng., Pt. F—Radar and Sonar Navig.*, vol. 141, pp. 133–136, 1994.