Exact Repair for Distributed Storage Systems: Partial Characterization via New Bounds

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***Abstract*—The exact-repair problem for distributed storage systems is considered. Characterizing the optimal storage-vs-repair bandwidth tradeoff for such systems remains an open problem for more than four storage nodes. A new family of information theoretic lower bounds is provided for the storage-vs-repair bandwidth tradeoff. The bound recovers Tians bound for the $(4, 3, 3)$ system, and hence suffices for exact characterization for this system. Moreover, the bound improves upon the existing lower bounds for the $(5, 4, 4)$ system and also characterizes the partial boundary of optimal exact repair tradeoff.**

I. Introduction

Contemporary distributed storage systems store massive amounts of data over a set of distributed nodes. Besides the traditional goals of achieving reliability by introducing redundancy, new aspects such as efficient repair of failed storage nodes are becoming increasingly important. To address these issues, the concept of regenerating codes for distributed storage systems was introduced by Dimakis *et al.* [1]. A distributed storage system (DSS) consists of n storage nodes each with a storage capacity of α units of data such that the entire file of size $\mathcal{B}$ can be recovered by accessing any $k < n$ nodes. This is called as the reconstruction property of the DSS. Whenever a node fails, d nodes (where $k \leq d \leq n-1$) participate in the repair process by sending β units of data each. This procedure is termed as the regeneration of a failed node and β is referred to as the per-node repair bandwidth. In [1], it was shown that the maximum amount of data, F, that any regenerating code can store satisfies

$$F \leq \sum_{i=0}^{k-1} \min\left(\alpha, (d-i)\beta\right). \tag{1}$$

Thus, in order to store data of size F, there exists a fundamental tradeoff between α (storage) and $d\beta$ (total repair bandwidth). It was also shown in [1] that the above tradeoff is achievable for functional repair, which does not require the contents of the repaired node to be the same as the original node.

In contrast to functional repair, exact repair requires that the contents of the failed node must match with those stored in the original node. Exact repair is a practically appealing property specially when it is desirable that the stored contents remain intact over time. Furthermore, the file recovery process is also easier in this case as the reconstruction procedure need not change whenever a failed node is replaced. While characterizing the storage-vs-bandwidth tradeoff for the case of exact repair remains a challenging open problem in general, two extreme points of this tradeoff namely, the minimum storage regenerating case (MSR) and the minimum bandwidth regenerating (MBR) case have been studied extensively (see [3], [4] and references therein). Other notable works on code constructions beyond MSR and MBR points include [8], [9].

Tian has recently characterized the exact repair tradeoff for the $(4, 3, 3)$-DSS [5]. This result, which is based on a novel computer-aided approach showed that functional and exact repair problems are fundamentally different. Despite its originality, the solution involved solving an optimization problem which included 65535 variables and 1966112 constraints, making this approach non-scalable and intractable for larger system parameters. Moreover, such a computer-aided approach does not necessarily lead to intuition and insights which could be used to understand the exact repair problem for a general set of parameters. Notably, Sasidharan et.al in [7] brought some intuition in this regard and presented a simpler proof for the $(4, 3, 3)$ problem and also presented new bounds for the $(5, 4, 4)$-DSS.

In this paper, we present a new and general approach for obtaining information theoretic upper bounds on F for the exact repair problem. We first explain the key ideas through a simple proof for the $(4, 3, 3)$ problem recovering Tian's result. We next consider the $(5, 4, 4)$-DSS and obtain new bounds for its exact repair tradeoff. Through our general bounding framework, we recover the bound in [7]. More interestingly, we obtain a new bound which together with the achievability results in [9], characterize the optimal exact-repair tradeoff for $\alpha \geq 2\beta$.

II. problem Statement and Result

Notation: We use $[i] = \{1, 2, \ldots, i\}$ to denote the set of positive integeres not exceeding i, and denote the set of all node indices $N = [n] = \{1, 2, \ldots, n\}$. We use W_i to denote the content stored in node i, and extend this definition to $W_A = \{W_i; i \in A\}$ for any $A \subseteq N$. The repair data from i to j is shown by S_i^j. Note that since $n = d+1$, there is unique way to repair any node i, and thus this definition is well-defined.

We also set S_i^i, to be dummy variable with zero entropy, for consistency. Moreover, $S_A^B = \{S_i^j : i \in A, j \in B\}$.

Next we describe the exact repair problem and the associated constraints. An exact repair distributed storage system with parameters (n, k, d) and (α, β) is defined as follows. A DSS including a total of n storage device, each with capacity α. Some Data will be encoded, partitioned and stored on this system in a distributed fashion, such that following hold.

- MDS Property (data recovery)
  - Data can be deterministically recovered from the content of any k nodes: $H(\text{Data}|W_A) = 0$ for any $A \subseteq N$ satisfying $|A| \geq k$.
- Repairability Requirements
  - The content of any failed node can be recovered (repaired/duplicated) by receiving no more that β units of repair data from any other d nodes, that is, $H(W_i|S_A^i) = 0$ for any $A \subseteq N \setminus \{i\}$, with $|A| \geq d$, where $H(S_j^i) \leq \beta$ and $H(S_j^i|W_i) = 0$.

We next present the main result of this paper which is a new set of lower bounds on the exact repair tradeoff for the $(5, 4, 4)$ distributed storage system.

Theorem 1. *The exact repair capacity of an* $(n, k, d) = (5, 4, 4)$ *distributed storage system with per node storage* α *and total repair bandwidth* $d\beta$ *satisfies*

$$\begin{aligned} F &\leq 4\alpha \\ F &\leq 3\alpha + \beta \\ 3F &\leq 7\alpha + 6\beta \\ 3F &\leq 5\alpha + 10\beta \\ F &\leq 10\beta. \end{aligned}$$

For $(5, 4, 4)$-DSS, the functional repair-tradeoff is given by

$$F \leq \min(\alpha, 4\beta) + \min(\alpha, 3\beta) + \min(\alpha, 2\beta) + \min(\alpha, \beta).$$

Hence, the inequalities $F \leq 4\alpha$, $F \leq 3\alpha + \beta$, and $F \leq 10\beta$ follow directly from this bound. The bounds $3F \leq 7\alpha + 6\beta$ and $3F \leq 5\alpha + 10\beta$ demarcate the exact repair tradeoff from the functional repair tradeoff for this problem. We note here that the bound $3F \leq 7\alpha + 6\beta$ was also obtained in [7], however, our mechanism for obtaining this bound is completely different. Perhaps the most interesting aspect of Theorem 1 is the bound $3F \leq 5\alpha + 10\beta$, which together with a recent scheme of [9] **leads to the optimal characterization of the exact-repair tradeoff** for $\alpha \geq 2\beta$. All of these bounds, together with the achievable points resulting from the best-known code constructions are shown in Fig. 1.

In the next section, we present a general bounding mechanism which is used to prove Theorem 1 by showing the new inequalities: $3F \leq 7\alpha + 6\beta$ and $3F \leq 5\alpha + 10\beta$. We remark here that the key novelty is the bounding mechanism, which is completely general (i.e., not specific to the $(5, 4, 4)$ problem) and can be readily applied to the exact-repair problem with more general parameters.

We remark here that the proposed bounding technique leads to a simple proof of the exact-repair tradeoff for the $(4, 3, 3)$ DSS, which is presented in Section IV.

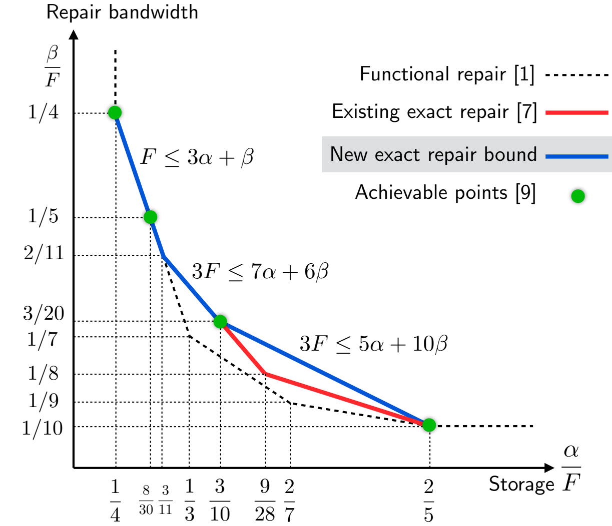

Fig. 1. Existing and new results for $(5, 4, 4)$ DSS.

III. Proof of Theorem 1

For the sake of brevity and simplicity, we focus on the tradeoff of the *symmetric*[1] exact-repair regeneration codes for distributed storage systems, in which the information-theoretical quantities are invariant under any relabeling of the nodes. We adopt the notation in [5] in order to formally define this symmetry:

Definition 1. *A* permutation π *is a one-to-one mapping*

$$\pi : N \to N,$$

where $N = \{1, 2, 3, 4, 5\}$. *We denote the set of all permutations by* Π.

Then a symmetric DSS can be defined as the following.

Definition 2. *An* (n, k, d) *exact-repair regeneration is called symmetric if for any subset of disk contents* $A \subseteq \{W_i : i = 1, 2, \ldots, 5\}$ *and repair data* $B \subseteq \{S_i^j : i, j = 1, 2, \ldots, 5\}$ *and permutation* $\pi \in \Pi$,

$$H(A, B) = H(\pi(A), \pi(B)).$$

We next present the statement and proof of lemma, which plays a central role in the proofs of the outer bounds. To this end, let $N = \{1, 2, 3, 4, 5\}$ denote the set of nodes in the system.

Lemma 1. *For any subsets of nodes* $A, B, C \subseteq \{1, 2, 3, 4, 5\}$, *we have*

$$H(S_N^{A \cup B}|W_C) + H(S_A^B, S_B^A|W_C) \leq H(S_N^A|W_C) + H(S_N^B|W_C).$$

[1] We note here that the symmetry assumption is made without any loss in generality, as any asymmetric code can be symmetrized by augmenting its $n!$ copies, each copy corresponding to a permutation of the node labels. The resulting symmetric code and the original asymmetric code would achieve the same (F, α, β) upto the scaling factor of $n!$.

Proof. First note that S_N^A includes S_B^A since $B \subseteq N$. Moreover, S_N^A provides all repair data required to repair nodes in A. Hence W_A can be reconstructed from S_N^A, from which the outgoing repair data S_A^B can be found. Hence

$$H(S_N^A|W_C) = H(S_N^A, S_A^B, S_B^A|W_C), \tag{2}$$

and similarly

$$H(S_N^B|W_C) = H(S_N^A, S_A^B, S_B^A|W_C). \tag{3}$$

Therefore, using the inequality

$$H(X,Y|T) + H(X,Z|T) \geq H(X,Y,Z|T) + H(X|T),$$

we get

$$\begin{aligned} & H(S_N^A|W_C) + H(S_N^B|W_C) \\ &\quad = H(S_N^A, S_A^B, S_B^A|W_C) + H(S_N^A, S_A^B, S_B^A|W_C) \\ &\quad \geq H(S_N^{A\cup B}, S_A^B, S_B^A|W_C) + H(S_A^B, S_B^A|W_C) \\ &\quad = H(S_N^{A\cup B}|W_C) + H(S_A^B, S_B^A|W_C), \end{aligned} \tag{4}$$

which implies the desired inequality. □

Having this lemma, we are ready to present the proof for the two inequality of interest.

A. *Proof of* $3F \leq 5\alpha + 10\beta$

Due to the fact that Data can be recovered from the content stored on any $k = 4$ nodes, we have

$$\begin{aligned} F = H(\mathsf{Data}) &= H(W_1, W_2, W_3, W_4) \\ &= \sum_{i=1}^{4} H\left(W_i|W_{[i-1]}\right) \\ &= \sum_{i=1}^{4}[H(W_i) - I\left(W_i; W_{[i-1]}\right)] \\ &\leq 4\alpha - \sum_{i=2}^{4} I\left(W_i; W_{[i-1]}\right). \end{aligned} \tag{5}$$

In order to further bound the data size F, we need to lower bound the summation $\sum_{i=2}^{4} I\left(W_i; W_{[i-1]}\right)$. First note that since the outgoing repair data from each node is a deterministic function of its content, we can write

$$\begin{aligned} \sum_{i=2}^{4} I\left(W_i; W_{[i-1]}\right) &\geq \sum_{i=1}^{4} I\left(S_i^{[i-1]}; S_{[i-1]}^i\right) \\ &= \sum_{i=2}^{4}\left[H\left(S_i^{[i-1]}\right) + H\left(S_{[i-1]}^i\right) - H\left(S_i^{[i-1]}, S_{[i-1]}^i\right)\right] \\ &= \underbrace{\sum_{i=2}^{4} H\left(S_i^{[i-1]}\right)}_{\mathsf{Term}_1} + \underbrace{\sum_{i=2}^{4} H\left(S_{[i-1]}^i\right)}_{\mathsf{Term}_2} \\ &\quad - \underbrace{\sum_{i=2}^{4} H\left(S_i^{[i-1]}, S_{[i-1]}^i\right)}_{\mathsf{Term}_3}. \end{aligned} \tag{6}$$

Hence, we can write

$$\sum_{i=2}^{4} I\left(W_i; W_{[i-1]}\right) \geq \mathsf{Term}_1 + \mathsf{Term}_2 - \mathsf{Term}_3. \tag{7}$$

Next we individually bound each of the terms in (6). In order to bound the first term, we can write

$$\begin{aligned} & \mathsf{Term}_1 + H(S_5^{1234}) \\ &\quad = H(S_2^1) + H(S_3^{12}) + H(S_4^{123}) + H(S_5^{1234}) \\ &\quad \geq H(S_2^1, S_3^{12}, S_4^{123}, S_5^{1234}) \\ &\quad = H(S_{2345}^1, S_{345}^2, S_{45}^3, S_5^4) \\ &\quad = H(S_{2345}^1, W_1, S_1^{234}, S_{345}^2, S_{45}^3, S_5^4) \\ &\quad \geq H(W_1, S_{1345}^2, W_2, S_2^{34}, S_{145}^3, S_{15}^4) \\ &\quad \geq H(W_1, W_2, S_{1245}^3, W_3, S_3^4, S_{125}^4) \\ &\quad \geq H(W_1, W_2, W_3, S_{1235}^4) \\ &\quad \geq H(W_1, W_2, W_3, W_4) = H(\mathsf{Data}) = F, \end{aligned} \tag{8}$$

which implies that

$$\begin{aligned} \mathsf{Term}_1 &= \sum_{i=2}^{4} H\left(S_i^{[i-1]}\right) \\ &\geq F - H\left(S_5^{1234}\right) \\ &\geq F - H(W_5) \geq F - \alpha. \end{aligned} \tag{9}$$

Moreover, each term in the summation on Term_2 in (6) can be lower bounded as follows. For $i = 2$, we have

$$\begin{aligned} H(S_1^2) &= \frac{1}{4}\left[H(S_1^2) + H(S_3^2) + H(S_4^2) + H(S_5^2)\right] \\ &\geq \frac{1}{4} H(S_{1345}^2) = \frac{1}{4} H(S_N^1). \end{aligned} \tag{10}$$

Similarly, for $i = 3$, we get

$$\begin{aligned} H(S_{12}^3) &= \frac{1}{6}\Big[H(S_{12}^3) + H(S_{14}^3) + H(S_{15}^3) \\ &\qquad + H(S_{24}^3) + H(S_{25}^3) + H(S_{45}^3)\Big] \\ &\overset{(*)}{\geq} \frac{3}{6} H(S_{1245}^3) = \frac{1}{2} H(S_N^1). \end{aligned} \tag{11}$$

where in $(*)$ we have used Han's inequality [6]. Lastly, for $i = 3$ we have

$$\begin{aligned} H(S_{123}^4) &= \frac{1}{4}\left[H(S_{123}^4) + H(S_{125}^4) + H(S_{135}^4) + H(S_{235}^4\right] \\ &\overset{(*)}{\geq} \frac{3}{4} H(S_{1235}^4) = \frac{3}{4} H(S_N^1). \end{aligned} \tag{12}$$

Hence, we have

$$\begin{aligned} \mathsf{Term}_2 &= H(S_1^2) + H(S_{12}^3) + H(S_{123}^4) \\ &\geq \frac{3}{2} H(S_N^1). \end{aligned} \tag{13}$$

Finally Term_3 in (6) can be bound using Lemma 1. Applying the lemma three times for $C = \emptyset$, and

a) $(A, B) = (\{1\}, \{2\})$:

$$H(S_N^{12}) + H(S_1^2, S_2^1) \leq H(S_N^1) + H(S_N^2). \tag{14}$$

b) $(A,B)=(\{1,2\},\{3\})$:

$$H(S_N^{123})+H(S_{12}^3,S_3^{12})\leq H(S_N^{12})+H(S_N^3). \quad (15)$$

c) $(A,B)=(\{1,2,3\},\{4\})$:

$$H(S_N^{1234})+H(S_{123}^4,S_4^{123})\leq H(S_N^{123})+H(S_N^4). \quad (16)$$

We can add up (14)-(16), to obtain

$$\begin{aligned}\mathsf{Term}_3 &= H(S_1^2,S_2^1)+H(S_{12}^3,S_3^{12})+H(S_{123}^4,S_4^{123})\\ &\leq H(S_N^1)+H(S_N^2)+H(S_N^3)+H(S_N^4)-H(S_N^{1234})\\ &= 4H(S_N^1)-F. \quad (17)\end{aligned}$$

Substituting (9), (13), and (17) into (6), we get

$$\begin{aligned}&\sum_{i=2}^{4} I\left(W_i;W_{[i-1]}\right)\\ &\quad\geq \sum_{i=2}^{4}\left[H\left(S_i^{[i-1]}\right)+H\left(S_{[i-1]}^i\right)-H\left(S_i^{[i-1]},S_{[i-1]}^i\right)\right]\\ &\quad= \mathsf{Term}_1+\mathsf{Term}_2-\mathsf{Term}_3\\ &\quad\geq [F-\alpha]+\frac{3}{2}H(S_N^1)-\left[4H(S_N^i-F)\right]\\ &\quad= 2F-\alpha-\frac{5}{2}H(S_N^1), \quad (18)\end{aligned}$$

which together with (5) implies

$$F\leq 4\alpha-\left[2F-\alpha-\frac{5}{2}H(S_N^1)\right]. \quad (19)$$

Note that $H(S_N^1)=H(S_2^1,S_3^1,S_4^1,S_5^1)\leq 4\beta$. Therefore, by simplifying this bound, we get the first claimed result

$$3F\leq 5\alpha+10\beta. \quad (20)$$

B. Proof of $3F\leq 7\alpha+6\beta$

The essence of proof here is similar to that presented in the proof of the previous section. The main distinction is that from the very first step we condition all the entropies and mutual information terms on W_5, the content of the fifth node.

$$\begin{aligned}F &= H(\mathsf{Data})=H(W_1,W_2,W_3,W_5)\\ &= H(W_5)+H(W_1,W_2,W_3|W_5)\\ &= H(W_5)+\sum_{i=1}^{3}H(W_i|W_{[i-1]},W_5)\\ &= H(W_5)+\sum_{i=1}^{3}[H(W_i|W_5)-I(W_i;W_{[i-1]}|W_5)\\ &\leq H(W_5)+\sum_{i=1}^{3}H(W_i|W_5)-\sum_{i=2}^{3}I(W_i;W_{[i-1]}|W_5)\\ &\leq 4\alpha-\sum_{i=2}^{3}I(W_i;W_{[i-1]}|W_5). \quad (21)\end{aligned}$$

Next, we have

$$\begin{aligned}&\sum_{i=2}^{3}I(W_i;W_{[i-1]}|W_5)\geq\sum_{i=2}^{3}I\left(S_i^{[i-1]};S_{[i-1}^i|W_5\right)\\ &=\sum_{i=2}^{3}\Big[H\Big(S_i^{[i-1]}|W_5\Big)+H\left(S_{[i-1]}^i|W_5\right)-H\Big(S_i^{[i-1]},S_{[i-1]}^i|W_5\Big)\Big]\\ &=\underbrace{\sum_{i=2}^{3}H\left(S_i^{[i-1]}|W_5\right)}_{\mathsf{Term}_1}+\underbrace{\sum_{i=2}^{3}H\left(S_{[i-1]}^i|W_5\right)}_{\mathsf{Term}_2}\\ &\quad-\underbrace{\sum_{i=2}^{3}H\left(S_i^{[i-1]},S_{[i-1]}^i|W_5\right)}_{\mathsf{Term}_3}. \quad (22)\end{aligned}$$

We can bound each term in (22), separately.

$$\begin{aligned}&\mathsf{Term}_1+H(S_4^{123}|W_5)\\ &\quad= H(S_2^1|W_5)+H(S_3^{12}|W_5)+H(S_4^{123}|W_5)\\ &\quad\geq H(S_2^1,S_3^{12}S_4^{123}|W_5)\\ &\quad= H(S_{2345}^1,S_{345}^2,S_{45}^3,S_5^4||W_5)\\ &\quad\geq H(W_1,W_2,W_3,W_4|W_5) \quad (23)\\ &\quad= H(\mathsf{Data}|W_5)\geq F-\alpha. \quad (24)\end{aligned}$$

where (23) follows from a similar argument as in (8).

Next, we use the symmetry of the problem, which further implies symmetry of the information-theoretical quantities, and write

$$\begin{aligned}H(S_1^2|W_5) &= \frac{1}{3}\left[H(S_1^2|W_5)+H(S_3^2|W_5)+H(S_4^2|W_5)\right]\\ &\geq\frac{1}{3}H(S_{134}^2|W_5)=\frac{1}{3}H(S_N^1|W_5). \quad (25)\end{aligned}$$

Similarly,

$$\begin{aligned}H(S_{12}^3|W_5) &= \frac{1}{3}\left[H(S_{12}^3|W_5)+H(S_{14}^3|W_5)+H(S_{24}^3|W_5)\right]\\ &\overset{(*)}{\geq}\frac{2}{3}H(S_{124}^3|W_5)=\frac{2}{3}H(S_N^1|W_5), \quad (26)\end{aligned}$$

where again inequality in $(*)$ is due to the conditional version of Han's inequality [6]. Thus, (25) along with (26) provides a lower bound on Term_2 in (22).

Additionally, in order to bound Term_3 in (22), we applying Lemma 1 with $C=\{5\}$ to

a) $(A,B)=(\{1\},\{2\})$:

$$\begin{aligned}&H(S_N^{12}|W_5)+H(S_2^1,S_1^2|W_5)\\ &\qquad\leq H(S_N^1|W_5)+H(S_N^1|W_5). \quad (27)\end{aligned}$$

b) $(A,B)=(\{1,2\},\{3\})$:

$$\begin{aligned}&H(S_N^{123}|W_5)+H(S_{12}^3,S_3^{12}|W_5)\\ &\qquad\leq H(S_N^{12}|W_5)+H(S_N^3|W_5). \quad (28)\end{aligned}$$

By summing up (27) and (28) we obtain

$$\begin{aligned}\mathsf{Term}_3 &= H(S_2^1, S_1^2|W_5) + H(S_{12}^3, S_3^{12}|W_5)\\ &\le \sum\nolimits_{i=1}^{3} H(S_N^i|W_5) - H(S_N^{123}|W_5)\\ &= 3H(S_N^1|W_5) - H(W_1,W_2,W_3,W_5|W_5)\\ &= 3H(S_N^1|W_5) - H(\mathsf{Data}|W_5)\\ &= 3H(S_N^1|W_5) - F + \alpha \qquad (29)\end{aligned}$$

Finally, we substitute (24), (25), (26), and (29) into (22) to get

$$\begin{aligned}&\sum\nolimits_{i=2}^{3} I(W_i; W_{[i-1]}|W_5)\\ &\quad\ge \mathsf{Term}_1 + \mathsf{Term}_2 - \mathsf{Term}_3\\ &\quad\ge \left[F - \alpha - H(S_4^{123}|W_5)\right] + H(S_N^1|W_5)\\ &\qquad - \left[3H(S_N^1|W_5) - F + \alpha\right]\\ &\quad= 2F - 2\alpha - H(S_4^{123}|W_5) - 2H(S_N^1|W_5), \qquad (30)\end{aligned}$$

which together with (21) implies

$$F \le 4\alpha - \left[2F - 2\alpha - H(S_4^{123}|W_5) - 2H(S_N^1|W_5)\right]. \qquad (31)$$

Again note that $H(S_N^1|W_5) = H(S_2^1, S_3^1, S_4^1|W_5) \le 3\beta$. On the other hand, we have

$$H(S_4^{123}|W_5) \le H(W_4|W_5) \le H(W_4) \le \alpha. \qquad (32)$$

Hence, using these inequalities, we have the proof for

$$3F \le 7\alpha + 6\beta. \qquad (33)$$

IV. Proof of $3F \le 4\alpha + 6\beta$ Bound for $(4,3,3)$ DSS

In this section we show that the machinery proposed in this paper to obtain upper bounds on the optimum tradeff of exact repair DSS can be applied to a wide range of system peremeters. In particular, we demonstrate how the novel bound of [5] for $(4,3,3)$ system can be subsumed from a similar argument.

Consider the $(4,3,3)$ exact-repair system, and let $N = [4] = \{1,2,3,4\}$. Similar to (5), we have

$$\begin{aligned}F &= H(\mathsf{Data}) = H(W_1,W_2,W_3) = \sum_{i=1}^{3} H\left(W_i|W_{[i-1]}\right)\\ &\le 3\alpha - \sum_{i=2}^{3} I\left(W_i; W_{[i-1]}\right) \le 3\alpha - \sum_{i=2}^{3} I\left(S_i^{[i-1]}; S_{[i-1]}^i\right)\\ &= 3\alpha - \underbrace{\sum_{i=2}^{3} H\left(S_i^{[i-1]}\right)}_{\mathsf{Term}_1} - \underbrace{\sum_{i=2}^{3} H\left(S_{[i-1]}^i\right)}_{\mathsf{Term}_2} + \underbrace{\sum_{i=2}^{3} H\left(S_i^{[i-1]}, S_{[i-1]}^i\right)}_{\mathsf{Term}_3}.\end{aligned}$$

The bounding techniques for Term_1, Term_2 and Term_3 follow along the same line of arguments as used for the $(5,4,4)$ DSS. More precisely, it can be readily show that

$$\mathsf{Term}_1 + H(S_4^{123}) \ge H(W_1,W_2,W_3) = F. \qquad (34)$$

Moreover, using Han's inequality, we have

$$H(S_1^2) \ge \frac{1}{3}H(S_N^1), \qquad \text{and} \qquad H(S_{12}^3) \ge \frac{2}{3}H(S_N^1)$$

which together imply

$$\mathsf{Term}_2 = \sum_{i=2}^{3} H\left(S_{[i-1]}^i\right) \ge H(S_N^1). \qquad (35)$$

Finally, we use Lemma 1 with $C = \emptyset$, for $(A,B) = (\{1\},\{2\})$ and $(A,B) = (\{1,2\},\{3\})$ to get

$$\begin{aligned}H(S_N^{12}) + H(S_1^2, S_2^1) &\le H(S_N^1) + H(S_N^2),\\ H(S_N^{123}) + H(S_{12}^3, S_3^{12}) &\le H(S_N^{12}) + H(S_N^3).\end{aligned}$$

These inequalities lead to

$$\mathsf{Term}_3 = \sum_{i=2}^{3} H\left(S_{[i-1]}^i, S_i^{[i-1]}\right) \le 3H(S_N^1) - F. \qquad (36)$$

Using the inequalities (34)-(36), we arrive at

$$\begin{aligned}F &\le 3\alpha - \mathsf{Term}_1 - \mathsf{Term}_2 + \mathsf{Term}_3\\ &\le 3\alpha - [F - H(S_4^{123})] - H(S_N^1) + [3H(S_N^1) - F], \qquad (37)\end{aligned}$$

which implies

$$3F \le 3\alpha + H(S_4^{123}) + 2H(S_N^1) \le 4\alpha + 6\beta. \qquad (38)$$

Note that the last inequality follows from the facts that $H(S_4^{123}) \le H(W_4) \le \alpha$, and $H(S_N^1) = H(S_2^1, S_3^1, S_4^1) \le 3\beta$. This concludes the proof of the bound for the $(4,3,3)$ DSS, highlighting the usefulness of the proposed bounding mechanism for exact-repair DSS.

V. Discussions

We studied the exact-repair problem for the $(5,4,4)$-distributed storage system and obtained new bounds on its optimum tradeoff. Our bounds indicate a gap between its functional and exact-repair tradeoffs for a wide range of parameters, namely $\alpha/\beta \in (3/2, 2)$. The novel bound is achievable for $\alpha \ge 2\beta$, which characterizes the optimum tradeoff for this regime.

On the other hand, for the regime $\alpha/\beta < 4/3$, functional and exact repair tradeoffs are the same and follow directly from existing results. However, for $\alpha/\beta \in (4/3, 2)$, there is a gap between the outer bound and the tradeoff one can achieve by space-sharing over the best known codes in this regime, corresponding to $(\alpha/F, \beta/F) = (4/15, 3/15)$ and $(\alpha/F, \beta/F) = (6/20, 3/20)$. We conjecture that space-sharing between these codes is indeed optimal for this regime, and the upper bound needs refinement. In particular, we believe that the bound in (32) is loose, and needs a more careful treatment.

While our results provide the state-of-the-art bounds for $(5,4,4)$ exact-repair distributed storage system, and partially characterizes its optimum tradeoff, the central contribution of this paper is to introduce a novel bounding mechanism and demonstrate its applicability in finding necessary conditions on the maximum system capacity. The advantage of the proposed approach is its applicability to the exact repair problem for a wide range of system parameters. We will present a general version of these bounding techniques and the resulting bounds in [13].

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